

Asymptotics of the Random-Coding Union Bound in Quasi-Static Fading Channels

Josep Font-Segura
Universitat Pompeu Fabra
josep.font@ieee.org

Alfonso Martinez
Universitat Pompeu Fabra
alfonso.martinez@ieee.org

Albert Guillén i Fàbregas
Universitat Pompeu Fabra
ICREA and University of Cambridge
guillen@ieee.org

Abstract—This paper studies the random-coding union (RCU) bound to the error probability in quasi-static fading channels. An asymptotic expansion and a normal approximation to the RCU bound suggest that the error probability converges to the outage probability as $\frac{1}{n}$, where n is the codeword blocklength. We particularize our results for Rayleigh fading, and compare them with the conventional normal approximation.

I. INTRODUCTION

Delay-constrained communication over slowly varying fading channels is characterized by outages. For most fading distributions, the channel capacity is zero, since there is an irreducible probability of error, i.e., the probability that the intended rate exceeds the instantaneous mutual information of the channel [1]–[3]. In terms of rates, fundamental limits are determined by the outage capacity, the largest achievable rate for a fixed outage probability. The convergence to the outage capacity was addressed in [4], [5] by studying the second-order coding rates. These works suggest that the achievable rates exhibit a backoff from the outage capacity that vanish faster than $\frac{1}{\sqrt{n}}$, where n is the codeword blocklength.

In terms of error probability, the bounds reported in [1], [3] are inadequate to study the convergence to the outage probability, since they are only asymptotically tight. We consider a weakened version of the random-coding union (RCU) bound [6], denoted as RCU_s . We propose an asymptotic expansion of the RCU_s bound in inverse powers of the blocklength, and a normal approximation that is a refined version of the conventional normal approximation [5], [6]. In particular, for a fixed rate R , we show that the RCU_s can be expanded as

$$\text{RCU}_s(n) = P_{\text{out}}(R) + \frac{\delta(R)}{n} + O\left(\frac{1}{n^2}\right), \quad (1)$$

where $P_{\text{out}}(R)$ is the outage probability, and $\delta(R)$ is a quantity that also depends on the rate.

II. CHANNEL MODEL

We consider the transmission of codewords of blocklength n over a quasi-static fading channel, where the channel gain H remains constant for the duration of the codeword and

This work has been funded in part by the European Research Council under ERC grant agreement 259663, by the European Union's 7th Framework Programme under grant agreement 303633 and by the Spanish Ministry of Economy and Competitiveness under grants RYC-2011-08150, TEC2012-38800-C03-03, and FJCI-2014-22747.

changes independently from codeword to codeword. The channel model is given by

$$\mathbf{Y} = H\mathbf{x} + \mathbf{Z}, \quad (2)$$

where $\mathbf{x} = (x_1, \dots, x_n)$ is the transmitted codeword, the noise \mathbf{Z} is independent of H , and has n independent and identically distributed (i.i.d.) $\mathcal{N}_{\mathbb{C}}(0, \sigma^2)$ entries. We consider a random channel gain H with unit variance and continuously differentiable probability density function $f_H(h)$. Codewords have an average power E_s , thus the signal-to-noise ratio of (2) is $\text{SNR} = \frac{E_s}{\sigma^2}$. Given a channel gain h , the transition probability during the transmission of \mathbf{x} is factorized as $W^n(\mathbf{y}|\mathbf{x}, h) = \prod_{i=1}^n W(y_i|x_i, h)$ where $W(y|x, h) = \frac{1}{\pi\sigma^2} e^{-\frac{1}{\sigma^2}|y-hx|^2}$. We assume that the receiver perfectly knows the realization of H .

The communication over quasi-static fading channel at a rate R is limited by the outage probability, defined as

$$P_{\text{out}}(R) = \mathbb{P}[I(H) < R] \quad (3)$$

where $I(h) = I(X; Y|H = h)$ is the instantaneous mutual information for a given channel realization. In the sequel, it will prove convenient to express the outage probability as the tail probability $P_{\text{out}}(R) = \mathbb{P}[\Phi_{\text{out}}(H) < 0]$ of the outage random variable $\Phi_{\text{out}}(h) = I(h) - R$. The outage random variable is characterized by the cumulant-generating function $\kappa_{\text{out}}(t) = \log \mathbb{E}[e^{-t\Phi_{\text{out}}(H)}]$ that can be related to the rate R and the instantaneous mutual information $I(h)$ through

$$\kappa_{\text{out}}(t) = tR + \log \mathbb{E}[e^{-tI(H)}]. \quad (4)$$

III. ERROR PROBABILITY

Random-coding arguments show that the error probability averaged over the code ensemble, denoted by $P_e(n)$, converges to the outage probability as n grows to infinity [1]–[3]. In this work, we study the second term of the asymptotic expansion of the error probability in inverse powers of n for the quasi-static memoryless fading channel with i.i.d. input distribution.

In particular, we study the RCU upper bound to the random-coding error probability $P_e(n)$ reported by [6]. Applying Markov's inequality, we weaken the RCU bound to [7]

$$\text{RCU}(n) \leq \mathbb{P}[\Phi_s(\mathbf{X}; \mathbf{Y}|H) \leq 0], \quad (5)$$

where $\Phi_s(\mathbf{x}; \mathbf{y}|h)$ is defined for some $s \geq 0$ as

$$\Phi_s(\mathbf{x}; \mathbf{y}|h) = i_s(\mathbf{x}; \mathbf{y}|h) + \log U - nR, \quad (6)$$

where U is uniformly distributed in $(0, 1)$, and $i_s(\mathbf{x}; \mathbf{y}|h)$ denotes the titled information density. For i.i.d. input distributions, $i_s(\mathbf{x}; \mathbf{y}|h) = \sum_{i=1}^n i_s(x_i; y_i|h)$ where

$$i_s(x; y|h) = \log \frac{W(y|x, h)^s}{\mathbb{E}[W(y|\bar{X}, h)^s]}. \quad (7)$$

In (7), x is distributed as P_X , and \bar{X} has the same distribution but is independent of y . We denote the r.h.s. of (5) as $\text{RCU}_s(n)$. Following the footsteps of [7], the tail probability (5) can be expressed in terms of the inverse Laplace transformation [8] as

$$\text{RCU}_s(n) = \frac{1}{2\pi j} \lim_{T \rightarrow \infty} \int_{\nu-jT}^{\nu+jT} \frac{e^{\kappa_s(t)}}{t} dt, \quad (8)$$

where $\kappa_s(t)$ is the cumulant-generating function of $\Phi_s(\mathbf{x}; \mathbf{y}|h)$, namely, $\kappa_s(t) = \log \mathbb{E}[e^{-t\Phi_s(\mathbf{X}; \mathbf{Y}|H)}]$. We assume that ν is within the region of convergence of (8). At this point, we choose $s^* = \frac{1}{1+t}$. Finally, for later convenience, we write $e^{\kappa_{s^*}(t)}$ in the following form

$$e^{\kappa_{s^*}(t)} = \frac{e^{\kappa(t)}}{1-t}, \quad (9)$$

where the term $\frac{1}{1-t}$ is the contribution of the random variable U , and $\kappa(t)$ is related to the information density (7) as

$$\kappa(t) = ntR + \log \mathbb{E} \left[\left(\mathbb{E} \left[e^{-ti_{s^*}(X; Y|H)} \middle| H \right] \right)^n \right]. \quad (10)$$

A. Asymptotic Expansion

Since we are interested in the behavior of the error probability as n increases, we can safely make the change of variable $nt = \alpha$ and integrate (8) over α , i.e.,

$$\text{RCU}_s(n) = \frac{1}{2\pi j} \lim_{T \rightarrow \infty} \int_{\nu-jT}^{\nu+jT} \frac{e^{\kappa(\frac{\alpha}{n})}}{\alpha(1-\frac{\alpha}{n})} d\alpha. \quad (11)$$

The region of convergence is $\nu \in (0, n)$ and $\kappa(\frac{\alpha}{n})$ can be written as

$$\kappa\left(\frac{\alpha}{n}\right) = \alpha R + \log \mathbb{E} \left[\left(\mathbb{E} \left[e^{-\frac{\alpha}{n} i_{s^*}(X; Y|H)} \middle| H \right] \right)^n \right]. \quad (12)$$

As $n \rightarrow \infty$, we make the following exponential expansions

$$\left(\mathbb{E} \left[e^{-\frac{\alpha}{n} i_{s^*}(X; Y|h)} \right] \right)^n = e^{-\alpha I(h) + \frac{\alpha^2}{2n} V(h)} + O\left(\frac{1}{n^2}\right), \quad (13)$$

where $I(h)$ and $V(h)$ are respectively the mean and variance of the tilted information spectrum $i_{s^*}(X; Y|h)$ for a given channel realization h , and $O\left(\frac{1}{n^2}\right)$ is a term that vanishes at least as fast as $\frac{1}{n^2}$ uniformly in α and h . We further note

$$\left(\mathbb{E} \left[e^{-\frac{\alpha}{n} i_{s^*}(X; Y|h)} \right] \right)^n = e^{-\alpha I(h)} \left(1 + \frac{\alpha^2}{2n} V(h) \right) + O\left(\frac{1}{n^2}\right). \quad (14)$$

Placing (14) into (12), and expanding the logarithm¹, we obtain that $\kappa\left(\frac{\alpha}{n}\right)$ can be expanded in inverse powers of the code blocklength n as

$$\kappa\left(\frac{\alpha}{n}\right) = \alpha R + \log \mathbb{E} \left[e^{-\alpha I(H)} \right] + \frac{1}{n} \frac{\alpha^2 \mathbb{E} \left[e^{-\alpha I(H)} V(H) \right]}{2 \mathbb{E} \left[e^{-\alpha I(H)} \right]} + O\left(\frac{1}{n^2}\right). \quad (15)$$

The two first terms of (16) correspond to the cumulant-generating function of the outage random variable, given in (4). Hence, the expansion of $\kappa\left(\frac{\alpha}{n}\right)$ can be expressed as

$$\kappa\left(\frac{\alpha}{n}\right) = \kappa_{\text{out}}(\alpha) + \frac{\kappa_1(\alpha)}{n} + O\left(\frac{1}{n^2}\right), \quad (16)$$

where we have defined $\kappa_1(\alpha)$ as the $\frac{1}{n}$ term of (15). Using (16) and further expanding remaining terms that depend on n in (11), we observe that

$$\frac{e^{\kappa(\frac{\alpha}{n})}}{\alpha(1-\frac{\alpha}{n})} = \frac{e^{\kappa_{\text{out}}(\alpha)}}{\alpha} \left(1 + \frac{\kappa_1(\alpha) + \alpha}{n} \right) + O\left(\frac{1}{n^2}\right) \quad (17)$$

$$= \frac{e^{\kappa_{\text{out}}(\alpha)}}{\alpha} + \frac{\theta(\alpha)}{n} + O\left(\frac{1}{n^2}\right). \quad (18)$$

Before looking into the details of $\theta(\alpha)$, we immediately see that the first term on the right-hand side of (18) is related to the tail probability of the outage random variable. More precisely, plugging (18) into (11), we obtain that the RCU_s can be expanded as

$$\text{RCU}_s(n) = P_{\text{out}}(R) + \frac{\delta(R)}{n} + O\left(\frac{1}{n^2}\right), \quad (19)$$

where we have defined $\delta(R)$ as the inverse Laplace transformation of $\theta(\alpha)$. The asymptotic expansion (19) suggests that the first term of the expansion of the error probability is indeed the outage probability, and that the second term of the expansion is a quantity that vanishes as $\frac{1}{n}$.

From (22), we observe that $\theta(\alpha)$ is composed of two terms, $\theta(\alpha) = \theta_1(\alpha) + \theta_2(\alpha)$. The first term is the contribution the pole $\alpha = n$ of (11), and is given by

$$\theta_1(\alpha) = e^{\kappa_{\text{out}}(\alpha)}. \quad (20)$$

On the other hand, the second term is the contribution of the expansion of $\kappa\left(\frac{\alpha}{n}\right)$ and is related to $I(h)$, $V(h)$, and R as

$$\theta_2(\alpha) = \frac{\alpha}{2} \mathbb{E} \left[e^{\alpha(R-I(H))} V(H) \right]. \quad (21)$$

Since the Laplace transformation is a linear operation, $\delta(R)$ can also be decomposed as $\delta(R) = \delta_1(R) + \delta_2(R)$, where

$$\delta_1(R) = \frac{1}{2\pi j} \lim_{T \rightarrow \infty} \int_{\nu-jT}^{\nu+jT} \theta_1(\alpha) d\alpha, \quad (22)$$

¹For two functions $f(X)$ and $g(X)$ of a random variable X satisfying the Lebesgue dominated convergence theorem as $n \rightarrow \infty$,

$$\log \mathbb{E} \left[f(X) \left(1 + \frac{g(X)}{n} \right) \right] = \log \mathbb{E}[f(X)] + \frac{\mathbb{E}[f(X)g(X)]}{n \mathbb{E}[f(X)]} + O\left(\frac{1}{n^2}\right).$$

and

$$\delta_2(R) = \frac{1}{2\pi j} \lim_{T \rightarrow \infty} \int_{\nu-jT}^{\nu+jT} \theta_2(\alpha) d\alpha. \quad (23)$$

Because $\theta_1(\alpha)$ is the moment-generating function of the outage random variable, $\delta_1(R)$ is the probability density function of Φ_{out} evaluated at the origin.

B. Gaussian Input over Rayleigh Fading

Under Rayleigh fading, $H \sim \mathcal{N}_{\mathbb{C}}(0, 1)$. The channel gain $Z = |H|^2$ has then an exponential probability distribution $f_Z(z) = e^{-z} \mathbb{1}\{z \geq 0\}$. For Gaussian input distribution, $I(h) = \log(1 + |h|^2 \text{SNR})$. Hence, the outage probability $P_{\text{out}}(R)$ and the cumulant-generating function $\kappa_{\text{out}}(t)$ are respectively given by

$$P_{\text{out}}(R) = 1 - \exp\left(-\frac{e^R - 1}{\text{SNR}}\right), \quad (24)$$

and

$$\kappa_{\text{out}}(t) = tR + \frac{1}{\text{SNR}} - t \log \text{SNR} + \log \Gamma\left(1 - t, \frac{1}{\text{SNR}}\right), \quad (25)$$

where $\Gamma(a, x) = \int_x^\infty t^{a-1} e^{-t} dt$ is the upper incomplete Gamma function. As noted above, $\delta_1(R)$ is the inverse Laplace transformation of the moment-generating function of the outage probability random variable $\Phi_{\text{out}}(h)$. Equivalently, $\delta_1(R)$ is the probability density function of $\Phi_{\text{out}}(h)$ sampled at the origin, i.e.,

$$\delta_1(R) = \frac{e^R}{\text{SNR}} \exp\left(-\frac{e^R - 1}{\text{SNR}}\right). \quad (26)$$

For i.i.d. Gaussian input distribution, the variance of the information density is given by [6]

$$V(h) = \frac{2|h|^2 \text{SNR}}{1 + |h|^2 \text{SNR}}. \quad (27)$$

Using $I(h)$ and $V(h)$ in (21), we obtain

$$\theta_2(\alpha) = \frac{e^{\alpha R + \frac{1}{\text{SNR}}}}{\text{SNR}^\alpha} \Gamma\left(1 - \alpha, \frac{1}{\text{SNR}}\right) \left(1 + \frac{1}{\text{SNR}} + \alpha\right) - \frac{e^{\alpha R}}{\text{SNR}}. \quad (28)$$

The former expression can be written in terms of the cumulant-generating function $\kappa_{\text{out}}(\alpha)$ as

$$\theta_2(\alpha) = e^{\kappa_{\text{out}}(\alpha)} \left(1 + \frac{1}{\text{SNR}} + \alpha\right) - \frac{e^{\alpha R}}{\text{SNR}}. \quad (29)$$

Similarly to the derivation of $\delta_1(R)$, the inverse Laplace transformation of $e^{\kappa_{\text{out}}(\alpha)}$ leads to the probability density function of $\Phi_{\text{out}}(h)$ evaluated at the origin. Secondly, the inverse Laplace transformation of $\alpha e^{\kappa_{\text{out}}(\alpha)}$ is the derivative of the probability density function of $\Phi_{\text{out}}(h)$ evaluated at the origin. Thirdly, $e^{\alpha R}$ is the moment-generating function of a random variable whose probability density function is a Dirac delta at $-R$. Therefore, the inverse Laplace transformation of $e^{\alpha R}$ is the probability density function of such random variable sampled at the origin, i.e., zero. Put together, we obtain

$$\delta_2(R) = \frac{e^R}{\text{SNR}^2} \exp\left(-\frac{e^R - 1}{\text{SNR}}\right) (1 - e^R + \text{SNR}). \quad (30)$$

This quantity was previously reported in [9] for uniform over the power sphere input distribution as $\delta_2(R) = -\frac{1}{2} f'_\Xi(0)$, where $f_\Xi(\xi)$ is the continuously differentiable probability density function of the random variable $\Xi = V(h)^{-\frac{1}{2}}(R - I(h))$.

Finally adding up (26) and (30), we find that under Rayleigh fading the RCU_s bound converges to the outage probability (24) as $\frac{1}{n} \delta(R)$, where

$$\delta(R) = \frac{e^R}{\text{SNR}^2} \exp\left(-\frac{e^R - 1}{\text{SNR}}\right) (1 - e^R + 2 \text{SNR}). \quad (31)$$

IV. NORMAL APPROXIMATION

We now propose an alternative approximation of the RCU_s bound. In particular, we approximate $(1 - \frac{\alpha}{n})^{-1}$ term of (11) as $(1 - \frac{\alpha}{n})^{-1} \approx e^{\frac{\alpha}{n}}$. Together with the relation (12) and the exponential expansion (14), we obtain that the RCU_s can be approximated as the inverse Laplace transformation of

$$\frac{e^{\kappa(\frac{\alpha}{n})}}{\alpha (1 - \frac{\alpha}{n})} \approx \frac{1}{\alpha} \mathbb{E} \left[e^{-\alpha [I(H) - R - \frac{1}{n}] + \frac{\alpha^2}{2n} V(H)} \right]. \quad (32)$$

This is $\frac{1}{\alpha}$ times the moment-generating function of a Gaussian random variable with mean $I(H) - R - \frac{1}{n}$ and variance $\frac{1}{n} V(H)$. We note that the $\frac{1}{n}$ term in the mean of this random variable is the contribution of the pole at $\alpha = n$. As a result, we may approximate the error probability by

$$\text{RCU}_s(n) \approx \mathbb{E} \left[Q \left(\frac{I(H) - R - \frac{1}{n}}{\sqrt{\frac{V(H)}{n}}} \right) \right]. \quad (33)$$

Since we can express the r.h.s. of (33) as the inverse Laplace transformation of the r.h.s. of (32), that is a first order expansion of (11), we use the derivations (11)–(23) to show that the normal approximation (33) can be also expanded as $P_{\text{out}}(R) + \frac{1}{n} \delta(R) + O(\frac{1}{n^2})$.

We note the similarity with the conventional normal approximation introduced in [5], [6], given by

$$P_e(n) \approx \mathbb{E} \left[Q \left(\frac{I(H) - R}{\sqrt{\frac{V(H)}{n}}} \right) \right]. \quad (34)$$

Since the contribution of the pole is not present, (34) will have a different $\frac{1}{n}$ term. We express (34) in terms of $\kappa(\frac{\alpha}{n})$. More specifically, we write (34) as

$$P_e(n) = \frac{1}{2\pi j} \lim_{T \rightarrow \infty} \int_{\nu-jT}^{\nu+jT} \frac{e^{\kappa(\frac{\alpha}{n})}}{\alpha} d\alpha, \quad (35)$$

with the approximation

$$e^{\kappa(\frac{\alpha}{n})} \approx \mathbb{E} \left[e^{-\alpha [I(H) - R] + \frac{\alpha^2}{2n} V(H)} \right]. \quad (36)$$

Hence, mimicking the derivations (11)–(23) with (35) in place of (11), we obtain that (34) can be asymptotically expanded in inverse powers of the blocklength n as

$$\mathbb{E} \left[Q \left(\frac{I(H) - R}{\sqrt{\frac{V(H)}{n}}} \right) \right] = P_{\text{out}}(R) + \frac{\delta_2(R)}{n} + O\left(\frac{1}{n^2}\right). \quad (37)$$

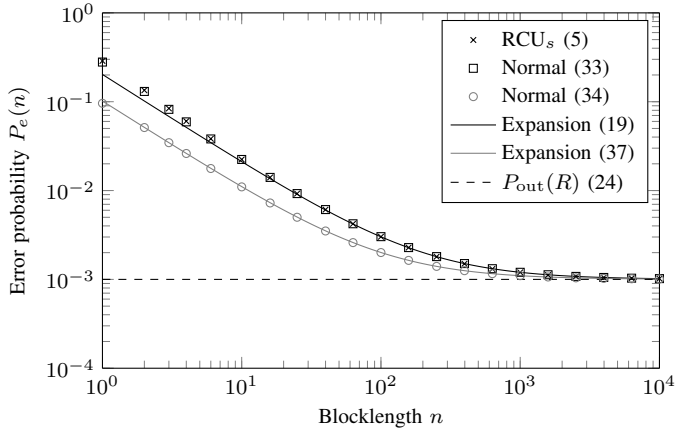


Fig. 1. Error probability versus blocklength for $R = 2$ and $\text{SNR} = 38$ dB.

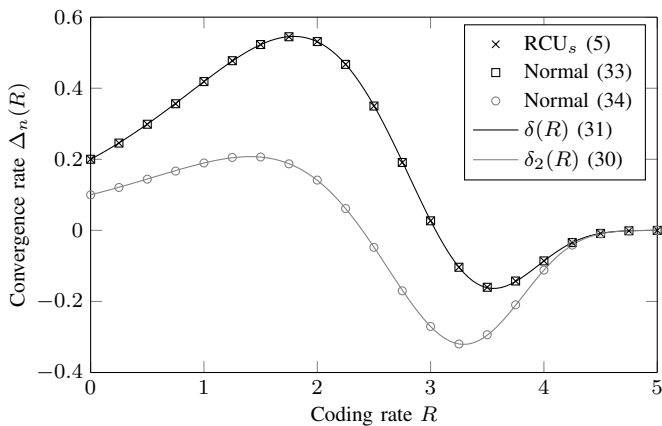


Fig. 2. Convergence rate versus rate R at $\text{SNR} = 10$ dB and $n = 10^4$.

In other words, the $\frac{1}{n}$ term of the expansion of (34) is $\delta_2(R)$ instead of $\delta(R)$. Since $\delta_1(R)$ is strictly positive, we have that $\delta(R) > \delta_2(R)$.

V. NUMERICAL RESULTS

We now evaluate the asymptotic expansions (19) and (37) with the simulated normal approximations (33), (34), and the simulated RCU_s (5) for Rayleigh fading and i.i.d. Gaussian input distribution.

In Fig. 1, we observe that the error probability converges to the outage probability $P_{\text{out}}(R)$. In this example, we have $P_{\text{out}}(R) = 10^{-3}$ for $R = 2$ and $\text{SNR} = 38$ dB. Comparing the RCU_s (5), the normal approximation (33), and the asymptotic expansion (19), we observe that the normal approximation and the asymptotic expansion are tight even for small blocklengths. This suggests that $\delta(R)$ is a dominant term of the error probability in quasi-static fading channels. On the other hand, the comparison between the normal approximation (34) and the Taylor expansion (37) shows that the conventional normal approximation of the information density exhibits a $\frac{1}{n}$ convergence to the outage probability proportional to $\delta_2(R)$, instead of $\delta(R)$.

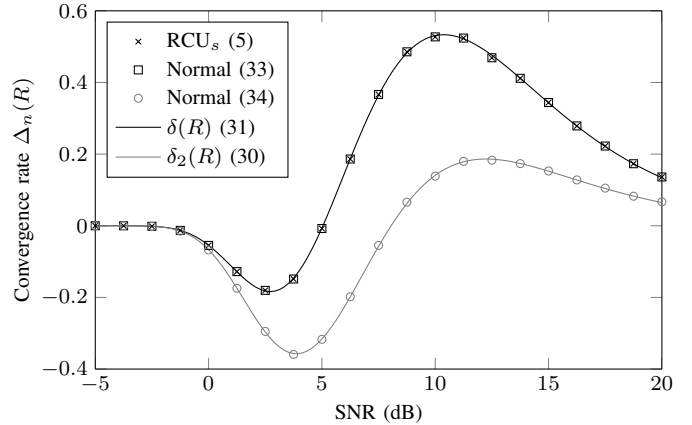


Fig. 3. Convergence rate versus SNR at $R = 2$ and $n = 10^4$.

To numerically evaluate the $\frac{1}{n}$ convergence to the outage probability, we define $\Delta_n(R)$ as

$$\Delta_n(R) = n(P_e(n) - P_{\text{out}}(R)), \quad (38)$$

where $P_e(n)$ acts as a placeholder for the approximations of Fig. 2 and Fig. 3. Both figures numerically validate the asymptotic expansions (19) and (37). That is, $\Delta_n(R) \rightarrow \delta(R)$ for $\text{RCU}_s(5)$ and for the normal approximation (33), whereas $\Delta_n(R) \rightarrow \delta_2(R)$ for the normal approximation (34). Hence, we focus on $\delta(R)$ and $\delta_2(R)$ versus R and SNR.

We note that the speed of convergence to the outage probability depends on the (R, SNR) pair. For a given SNR, the rate $R = \log(1 + 2\text{SNR})$ incurs a change of sign in $\delta(R)$. On the other hand, by fixing the rate R , $\delta(R)$ becomes strictly positive for $\text{SNR} > \frac{1}{2}(e^R - 1)$. We finally note that the same arguments hold for the $\frac{1}{n}$ convergence term of the expansion (37). Yet in this case, the zero crossings happen at a rate $R = \log(1 + \text{SNR})$, or equivalently $\text{SNR} = e^R - 1$.

REFERENCES

- [1] G. Kaplan and S. Shamai (Shitz), "Error probabilities for the block-fading Gaussian channel," *Archiv Elektro Übertragungstechnik (AEÜ)*, vol. 49, no. 4, pp. 192–205, 1995.
- [2] E. Biglieri, J. Proakis, and S. Shamai (Shitz), "Fading channels: information-theoretic and communications aspects," *IEEE Trans. Inf. Theory*, vol. 44, no. 6, pp. 2619–2692, Oct. 1998.
- [3] E. Malkamäki and H. Leib, "Coded diversity on block-fading channels," *IEEE Trans. Inf. Theory*, vol. 45, no. 2, pp. 771–781, Mar. 1999.
- [4] E. MolavianJazi and J. N. Laneman, "On the second-order coding rate of block fading channels," in *Proc. Allerton Conf. Commun., Contr., Comput.*, Oct. 2013.
- [5] W. Yang, G. Durisi, T. Koch, and Y. Polyanskiy, "Quasi-static multiple-antenna fading channels at finite blocklength," *IEEE Trans. Inf. Theory*, vol. 60, no. 7, pp. 4232–4265, 2014.
- [6] Y. Polyanskiy, H. Poor, and S. Verdú, "Channel coding rate in the finite blocklength regime," *IEEE Trans. Inf. Theory*, vol. 56, no. 5, pp. 2307–2359, May 2010.
- [7] A. Martinez, J. Scarlett, M. Dalai, and A. Guillén i Fàbregas, "A complex-integration approach to the saddlepoint approximation for random-coding bounds," in *Proc. Int. Symp. Wireless Commun. Systems (ISWCS)*, 2014, pp. 618–621.
- [8] G. Doetsch, *Introduction to the Theory and Applications of the Laplace Transformation*. Springer, 1974.
- [9] W. Yang, G. Durisi, T. Koch, and Y. Polyanskiy, "Quasi-static SIMO fading channels at finite blocklength," in *Proc. IEEE Int. Symp. Inf. Theory (ISIT)*, 2013, pp. 1531–1535.