Refined Error Probability Approximations in Quasi–Static Erasure Channels

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***Abstract*—This paper considers the transmission of codewords over a quasi–static binary erasure channel, where the erasure probability changes independently at each transmitted codeword. An approximation to the random–coding union bound suggests that the error probability exceeds the outage probability by a quantity that is inversely proportional to the blocklength.**

I. Introduction

A quasi–static channel is a good model for delay–constrained communication over slow–varying channels [1]. The outage capacity has been emphasized as the most important information–theoretic measure in quasi–static channels. However, little attention has been given to the error probability. In [2], the performance of the quasi–static fading channel is described by means of Gallager–type random–coding bounds. Malkamäki *et al.* [3] proposed a tighter bound, and showed that the average error probability is asymptotically given by the outage probability in the limit of infinite codeword blocklength [3, Th. 2]. However, for finite codeword blocklength, this tighter bound has to be evaluated numerically, as the optimization of the bound involves the fading coefficients.

This paper considers the random–coding union (RCU) bound [4] to the error probability in the simple quasi–static binary erasure channel (BEC). By writing the RCU bound as a tail probability, we propose two saddlepoint approximations [5] that build upon the techniques of [2], [3]. By inspecting the asymptotic behavior of the saddlepoint with the blocklength, we finally derive an expansion of the RCU bound in inverse powers of the blocklength that suggests that the error probability converges to the outage probability as $\frac{\delta(R)}{n}$, where n is the codeword blocklength, R is the rate of the code, and $\delta(R)$ is a rate–dependent constant.

II. Preliminaries

Consider the transmission of codewords of blocklength n symbols, where each codeword spans a single BEC with uniformly distributed erasure probability ε, that changes independently from codeword to codeword. Given the erasure probability ε, the transition probability during the transmission of a codeword can be factorized as

$$W_\varepsilon^n(\boldsymbol{y}|\boldsymbol{x}) = \prod_{i=1}^{n} W_\varepsilon(y_i|x_i), \tag{1}$$

where $\boldsymbol{x} = (x_1, \dots, x_n)$ and $\boldsymbol{y} = (y_1, \dots, y_n)$ are the channel input and channel output sequence, respectively, and $W_\varepsilon(y|x)$ denotes the transition probability of a single BEC of erasure probability ε [6].

We study the transmission of equiprobable messages $m \in \{1, \dots, M\}$, where each message is mapped onto a codeword $\boldsymbol{x}(m)$, and the collection of all codewords is a code of rate $R = \frac{1}{n}\log M$. For a fixed erasure probability ε, the average error probability of the code is denoted as $P_e(n, \varepsilon)$. Here, we are mostly interested in the error probability averaged over the erasure probability, i.e.,

$$P_e(n) = \mathbb{E}\left[P_e(n,\varepsilon)\right]. \tag{2}$$

Random–coding arguments show the existence of a code whose error probability is, at least, as good as that of the ensemble average. In this work, we consider such a code.

Two random–coding upper bounds to the error probability for the block–fading channel were reported by Malkamäki *et al.* in [3]. Particularized for the quasi–static BEC, the first bound is based on a conditional Gallager bound [7] given the erasure probability [3, Eq. (16)–(17)], i.e.,

$$P_e(n,\varepsilon) \le \begin{cases} 1 & \hat\rho_\varepsilon < 0 \\ e^{-n(E_0(\hat\rho_\varepsilon,\varepsilon) - \hat\rho_\varepsilon R)} & 0 \le \hat\rho_\varepsilon \le 1 \\ e^{-n(E_0(1,\varepsilon) - R)} & \hat\rho_\varepsilon > 1. \end{cases} \tag{3}$$

Then, the average over the erasure probability is applied. In (3), $\hat\rho_\varepsilon$ is the argument that maximizes $E_0(\rho,\varepsilon) - \rho R$, closely related to (15) later derived in the paper. As $\hat\rho_\varepsilon$ is a function of the erasure probability, the expectation of (3) with respect to ε has to be numerically evaluated for a finite blocklength. Asymptotically, the Gallager bound (3) shows that the error probability converges to the outage probability, denoted as $P_{\text{out}}(R)$ and given as

$$P_{\text{out}}(R) = \mathbb{P}\left[I(\varepsilon) < R\right], \tag{4}$$

where $I(\varepsilon)$ is the mutual information of a single BEC with erasure probability ε maximized over the input distribution.

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For the quasi–static BEC with uniformly distributed error probability, we have that

$$I(\varepsilon) = (1-\varepsilon)\log 2, \tag{5}$$

$$P_{\text{out}}(R) = \frac{R}{\log 2}. \tag{6}$$

A simpler bound was also proposed in [3, Eq. (22)] by first averaging the erasure probability and then optimizing a parameter that does not depend on ε:

$$P_e(n) \leq \begin{cases} 1 & \hat{\rho} < 0 \\ e^{-n(E_0(\hat{\rho})-\hat{\rho}R)} & 0 \leq \hat{\rho} \leq 1 \\ e^{-n(E_0(1)-R)} & \hat{\rho} > 1. \end{cases} \tag{7}$$

Here, $\hat{\rho}$ is the argument that maximizes $E_0(\rho) - \rho R$, closely related to (30) later derived in the paper. As pointed out in [3], (7) is a weaker bound to the error probability, as $\hat{\rho}$ will be only optimal for some realizations of ε.

In summary, Gallager arguments lead to a tighter bound that needs to be numerically evaluated, and a simpler bound that is especially loose in quasi–static channels (see [3, Fig. 2]). In this work, we discuss whether the performance gap between (3) and (7) is a genuine issue of quasi–static channels by studying more refined expressions of the error probability based on the random–coding union bound. We further study the convergence of the error probability to the outage probability. For a different perspective, the dual problem, i.e., the convergence of the achievable rate to the outage capacity, see the recent work by Yang *et al.* [8].

III. Saddlepoint Approximations

A. Saddlepoint Approximation of RCU(n, ε)

For a fixed BEC realization of the erasure probability ε, the RCU bound to the average error probability [4] is given by

$$P_e(n,\varepsilon) \leq \mathbb{E}\left[\min\left\{1, M\mathbb{P}\left[W_\varepsilon^n(\boldsymbol{Y}|\overline{\boldsymbol{X}}) \geq W_\varepsilon^n(\boldsymbol{Y}|\boldsymbol{X}) | \boldsymbol{X},\boldsymbol{Y}\right]\right\}\right], \tag{8}$$

where $\boldsymbol{X}$, $\boldsymbol{Y}$ are the random variables for channel input and channel output sequences, respectively, and $\overline{\boldsymbol{X}}$ is distributed as $\boldsymbol{X}$ but independent of $\boldsymbol{Y}$. As noted in [9], we can apply Markov's inequality and weaken the RCU bound as

$$P_e(n,\varepsilon) \leq \text{RCU}(n,\varepsilon) \tag{9}$$

where RCU(n, ε) is the tail probability

$$\text{RCU}(n,\varepsilon) = \mathbb{P}\left[\Phi_n(\boldsymbol{X},\boldsymbol{Y},\varepsilon) \leq 0\right]. \tag{10}$$

In (10), the random variable $\Phi_n(\boldsymbol{X},\boldsymbol{Y},\varepsilon)$ is

$$\Phi_n(\boldsymbol{X},\boldsymbol{Y},\varepsilon) = \sum_{i=1}^{n} i_s(X_i,Y_i,\varepsilon) + \log U - nR, \tag{11}$$

where U is a uniform $(0, 1)$ random variable, and the symbol s–information density is defined as

$$i_s(X,Y,\varepsilon) = \log \frac{W_\varepsilon(Y|X)^s}{\mathbb{E}[W_\varepsilon(Y|\overline{X})^s|Y]}. \tag{12}$$

For the quasi–static BEC, we note that (12) is independent on s, and that the bounds (8) and (10) coincide [9].

As noted in [10], the tail probability (10) can be expressed in terms of the inverse Laplace transformation [11] as

$$\text{RCU}(n,\varepsilon) = \frac{1}{2\pi j}\int_{\nu-j\infty}^{\nu+j\infty} \frac{\mathbb{E}\left[e^{-t\Phi_n(\boldsymbol{X},\boldsymbol{Y},\varepsilon)}\right]}{t}\,\mathrm{d}t, \tag{13}$$

where we assume that ν is within the range of convergence, i.e., $\nu \in (0, 1)$. The evaluation of the expectation term in (13), using (11) and (12), leads to

$$\mathbb{E}\left[e^{-t\Phi_n(\boldsymbol{X},\boldsymbol{Y},\varepsilon)}\right] = \frac{e^{\kappa_{n,\varepsilon}(t)}}{1-t}, \tag{14}$$

where $\kappa_{n,\varepsilon}(t)$ is given as

$$\kappa_{n,\varepsilon}(t) = ntR + n\log\left(\frac{1}{2^t}(1-\varepsilon)+\varepsilon\right). \tag{15}$$

We note that the former expression can be written in terms of the Gallager function $E_0(t, \varepsilon)$ that appears in (3) through $\kappa_{n,\varepsilon}(t) = -n(E_0(t,\varepsilon) - tR)$. The critical points of (13) are two poles at $t = 0$ and $t = 1$, and a saddlepoint at $t = t_{n,\varepsilon}$, the absolute minimum of $\kappa_{n,\varepsilon}(t)$ in the real axis, i.e.,

$$t_{n,\varepsilon} = \underset{-\infty<t<\infty}{\arg\min}\,\kappa_{n,\varepsilon}(t). \tag{16}$$

If $0 \leq t_{n,\varepsilon} \leq 1$, it is safe to set $\nu = t_{n,\varepsilon}$ in (13). Yet, whenever $t_{n,\varepsilon} < 0$ and $t_{n,\varepsilon} > 1$, the poles at $t = 0$ and $t = 1$ introduce additional terms due to the Cauchy's residue theorem [11].

Since no closed–form solutions to the complex–integration (13) are available in general, we propose a Taylor expansion of $\kappa_{n,\varepsilon}(t)$ around $t_{n,\varepsilon}$, i.e.,

$$\kappa_{n,\varepsilon}(t) \approx \kappa_{n,\varepsilon}(t_{n,\varepsilon}) + \kappa'_{n,\varepsilon}(t_{n,\varepsilon})(t-t_{n,\varepsilon}) + \frac{1}{2}\kappa''_{n,\varepsilon}(t_{n,\varepsilon})(t-t_{n,\varepsilon})^2, \tag{17}$$

where $\kappa'_{n,\varepsilon}(t)$ and $\kappa''_{n,\varepsilon}(t)$ denote, respectively, the first and second derivatives of $\kappa_{n,\varepsilon}(t)$ with respect to t. Finally, following the footsteps of [10], we obtain that the RCU bound for the quasi–static BEC can be approximated as

$$\text{RCU}(n,\varepsilon) \approx \gamma_{n,\varepsilon} + \sigma_{n,\varepsilon}e^{\kappa_{n,\varepsilon}(t_{n,\varepsilon})}. \tag{18}$$

Here, the additive term $\gamma_{n,\varepsilon}$ can be expressed as

$$\gamma_{n,\varepsilon} = \begin{cases} 1 & t_{n,\varepsilon} < 0 \\ 0 & 0 \leq t_{n,\varepsilon} \leq 1 \\ e^{\kappa_{n,\varepsilon}(1)} & t_{n,\varepsilon} > 1, \end{cases} \tag{19}$$

whereas the pre–exponential term $\sigma_{n,\varepsilon}$ is given by

$$\sigma_{n,\varepsilon} = \overline{Q}\left(t_{n,\varepsilon}\sqrt{\kappa''_{n,\varepsilon}(t_{n,\varepsilon})}\right) + \overline{Q}\left((1-t_{n,\varepsilon})\sqrt{\kappa''_{n,\varepsilon}(t_{n,\varepsilon})}\right), \tag{20}$$

where

$$\overline{Q}(x) = \text{sign}(x)\frac{1}{2}\text{erfc}\left(\frac{|x|}{\sqrt{2}}\right)e^{\frac{x^2}{2}}. \tag{21}$$

The proposed approximation of the RCU involves determining the saddlepoint of (15), given by

$$t_{n,\varepsilon} = \log_2\left(\frac{1-\varepsilon}{\varepsilon}\frac{\log 2 - R}{R}\right). \tag{22}$$

It is straightforward to show that, asymptotically, the saddlepoint approximation (18) satisfies

$$\lim_{n\to\infty} \text{RCU}(n,\varepsilon) = \mathbb{1}\{I(\varepsilon) < R\}, \tag{23}$$

where $\mathbb{1}\{\cdot\}$ is the indicator function. As $n \to \infty$, the saddlepoint approximation (18) approaches a Bernoulli random variable with probability of success $P_{\text{out}}(R)$. Since this random variable is bounded, we can apply the Lebesgue dominated convergence theorem [12] to prove that

$$\lim_{n\to\infty} \mathbb{E}\left[\text{RCU}(n,\varepsilon)\right] = P_{\text{out}}(R). \tag{24}$$

Therefore, the average of the saddlepoint approximation to the RCU given the erasure probability, shows that the error probability converges to the outage probability, but gives no direct information about the rate of this convergence.

B. Saddlepoint Approximation of RCU(n)

For symmetry with the work by Malkamäki *et al.* [3], we now study the RCU bound to the error probability averaged over the erasure probability, i.e.,

$$\begin{aligned} P_e(n) \leq \mathbb{E}\big[\min\{1, \\ M\mathbb{P}\left[W_\varepsilon^n(\boldsymbol{Y}|\overline{\boldsymbol{X}}) \geq W_\varepsilon^n(\boldsymbol{Y}|\boldsymbol{X})|\boldsymbol{X},\boldsymbol{Y},\varepsilon\right]\}\big]. \end{aligned} \tag{25}$$

Similarly to (9), we can apply the Markov's inequality and further weaken (25) as

$$P_e(n) \leq \text{RCU}(n), \tag{26}$$

where now the erasure probability ε is treated as a random variable in the evaluation of the tail probability of $\Phi_n(\boldsymbol{X},\boldsymbol{Y},\varepsilon)$, i.e.,

$$\text{RCU}(n) = \mathbb{P}\left[\Phi_n(\boldsymbol{X},\boldsymbol{Y},\varepsilon) \leq 0\right]. \tag{27}$$

We can again express the tail probability (27) in terms of the inverse Laplace transformation as

$$\text{RCU}(n) = \frac{1}{2\pi j}\int_{\nu-j\infty}^{\nu+j\infty} \frac{\mathbb{E}\left[e^{-t\Phi_n(\boldsymbol{X},\boldsymbol{Y},\varepsilon)}\right]}{t}\,\mathrm{d}t, \tag{28}$$

where ν is within the region of convergence, i.e., $\nu \in (0,1)$. Taking into account the erasure probability ε in the following expectation

$$\mathbb{E}\left[e^{-t\Phi_n(\boldsymbol{X},\boldsymbol{Y},\varepsilon)}\right] = \frac{e^{\kappa_n(t)}}{1-t}, \tag{29}$$

now $\kappa_n(t)$ is defined as

$$\kappa_n(t) = ntR + \log\frac{2^t - 2^{-nt}}{(2^t-1)(n+1)}. \tag{30}$$

Again, (30) is related to the Gallager function $E_0(t)$ involved in (7) through $\kappa_n(t) = -n\left(E_0(t) - tR\right)$. The saddlepoint to RCU(n) is defined as the absolute minimum of $\kappa_n(t)$ over the real axis, i.e.,

$$t_n = \operatorname*{arg\,min}_{-\infty<t<\infty} \kappa_n(t). \tag{31}$$

Similarly to (17), we approximate (28) by expanding $\kappa_n(t)$ around t_n, and obtain that the averaged RCU bound for the quasi–static BEC can be approximated as

$$\text{RCU}(n) \approx \gamma_n + \sigma_n e^{\kappa_n(t_n)} \tag{32}$$

where now

$$\gamma_n = \begin{cases} 1 & t_n < 0 \\ 0 & 0 \leq t_n \leq 1 \\ e^{\kappa_n(1)} & t_n > 1, \end{cases} \tag{33}$$

and

$$\sigma_n = \overline{Q}\left(t_n\sqrt{\kappa_n''(t_n)}\right) + \overline{Q}\left((1-t_n)\sqrt{\kappa_n''(t_n)}\right). \tag{34}$$

Even though closed-form expressions for the saddlepoint t_n are not available in this case, we further investigate its relation to the outage probability by proposing a saddlepoint approximation to the outage probability.

C. Saddlepoint Approximation of $P_{\text{out}}(R)$

We note that the outage probability (4) can be seen as a tail probability of the random variable

$$\Phi_{\text{out}}(R) = I(\varepsilon) - R. \tag{35}$$

Therefore, it is natural to express the outage probability as the inverse Laplace transformation

$$P_{\text{out}}(R) = \frac{1}{2\pi j}\int_{\nu-j\infty}^{\nu+j\infty} \frac{\mathbb{E}\left[e^{-t\Phi_{\text{out}}(R)}\right]}{t}\,\mathrm{d}t, \tag{36}$$

where now the expectation is only with respect to the erasure probability, and $\nu \in (0,\infty)$. This leads to

$$\mathbb{E}\left[e^{-t\Phi_{\text{out}}(R)}\right] = e^{\kappa_{\text{out}}(t)}, \tag{37}$$

where $\kappa_{\text{out}}(t)$ is given as

$$\kappa_{\text{out}}(t) = tR + \log\frac{1-2^{-t}}{t\log 2}. \tag{38}$$

Now, (36) has only one pole at $t = 0$ and a saddlepoint at

$$t_{\text{out}}(R) = \operatorname*{arg\,min}_{-\infty<t<\infty} \kappa_{\text{out}}(t). \tag{39}$$

Mimicking (17) with (38), we may hence approximate the outage probability as

$$P_{\text{out}}(R) \approx \gamma_{\text{out}}(R) + \sigma_{\text{out}}(R)e^{\kappa_{\text{out}}(t_{\text{out}}(R))}. \tag{40}$$

In this case, we have that the additive term $\gamma_{\text{out}}(R)$ and the pre–exponential term $\sigma_{\text{out}}(R)$ are given, respectively, as

$$\gamma_{\text{out}}(R) = \begin{cases} 1 & t_{\text{out}}(R) < 0 \\ 0 & t_{\text{out}}(R) \geq 0 \end{cases} \tag{41}$$

$$\sigma_{\text{out}}(R) = \overline{Q}\left(t_{\text{out}}(R)\sqrt{\kappa''(t_{\text{out}}(R))}\right). \tag{42}$$

IV. An Asymptotic Expansion of RCU(n)

One advantage of the complex–integration expression of the RCU (28) is that the average with respect to the erasure probability is naturally incorporated in the definition of $\kappa_n(t)$. By further inspecting the behavior of the saddlepoint to the RCU as the codeword blocklength $n \to \infty$, we numerically notice that the saddlepoint $t_n \to 0$. This motivates to study the behavior of the product nt_n, illustrated in Fig. 1 for three different rates. Remarkably, nt_n converges to $t_{\text{out}}(R)$. This suggests that it is safe to make the change of variable $nt = \alpha$ and integrate with α, i.e.,

$$\text{RCU}(n) = \frac{1}{2\pi j} \int_{\nu - j\infty}^{\nu + j\infty} \frac{e^{\kappa_n\left(\frac{\alpha}{n}\right)}}{\alpha\left(1 - \frac{\alpha}{n}\right)} \, \mathrm{d}\alpha, \tag{43}$$

where now the region of convergence is $\nu \in (0, n)$. From (30), we note that $\kappa_n\left(\frac{\alpha}{n}\right)$ has the form

$$\kappa_n\left(\frac{\alpha}{n}\right) = \alpha R + \log \frac{2^{\frac{\alpha}{n}} - 2^{-\alpha}}{(2^{\frac{\alpha}{n}} - 1)(n+1)}. \tag{44}$$

For sufficiently large codeword blocklength n, we derive a Taylor expansion in inverse powers of the codeword blocklength n, i.e.,

$$\frac{e^{\kappa_n\left(\frac{\alpha}{n}\right)}}{\alpha\left(1 - \frac{\alpha}{n}\right)} = \theta_0(\alpha) + \frac{\theta_1(\alpha)}{n} + O\left(\frac{1}{n^2}\right), \tag{45}$$

where $O\left(\frac{1}{n^2}\right)$ is a term that vanishes at least as fast as $\frac{1}{n^2}$, and the coefficients $\theta_0(\alpha)$ and $\theta_1(\alpha)$ are given by

$$\theta_0(\alpha) = \frac{e^{\alpha R}(1 - 2^{-\alpha})}{\alpha^2 \log 2}, \tag{46}$$

and

$$\theta_1(\alpha) = \frac{e^{\alpha R}(1 - 2^{-\alpha})}{\alpha \log 2} - \frac{e^{\alpha R}(1 - 2^{-\alpha})}{\alpha^2 \log 2} + \frac{e^{\alpha R}(1 + 2^{-\alpha})}{2\alpha}, \tag{47}$$

respectively. Comparing (38) and (46), we first observe that in fact $\theta_0(\alpha)$ is related to $\kappa_{\text{out}}(t)$ as

$$\theta_0(\alpha) = \frac{e^{\kappa_{\text{out}}(\alpha)}}{\alpha}. \tag{48}$$

Hence, we may identify $\theta_0(\alpha)$ with the evaluation of the outage probability

$$\frac{1}{2\pi j} \int_{\nu - j\infty}^{\nu + j\infty} \frac{e^{\alpha R}(1 - 2^{-\alpha})}{\alpha^2 \log 2} \, \mathrm{d}\alpha = P_{\text{out}}(R). \tag{49}$$

Regarding $\theta_1(\alpha)$, we identify that the first term of $\theta_1(\alpha)$ is actually $e^{\kappa_{\text{out}}(\alpha)}$, and therefore that the complex–integration of this term is the probability density function of Φ_{out} (35) evaluated at the origin (see [11]). Since ε is uniformly distributed, Φ_{out} is then uniformly distributed in the interval $[-R, \log 2 - R]$, and we have that

$$\frac{1}{2\pi j} \int_{\nu - j\infty}^{\nu + j\infty} \frac{e^{\alpha R}(1 - 2^{-\alpha})}{\alpha \log 2} \, \mathrm{d}\alpha = \frac{1}{\log 2}. \tag{50}$$

Likewise, we identify the second term of $\theta_1(\alpha)$ as $\theta_0(\alpha)$ in (48), again leading to the outage probability as in (49). Finally,

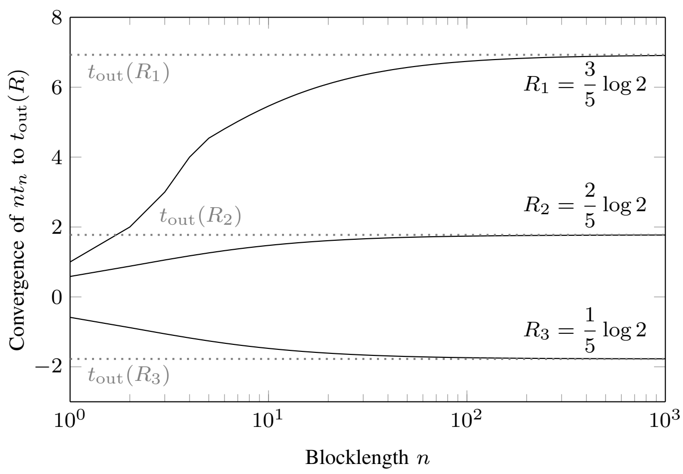

Fig. 1. Convergence of nt_n to $t_{\text{out}}(R)$, versus the blocklength n, for several rates.

the last term in (47) can be be split into two additive terms that are identified as tail probabilities of two random variables. The first one is a random variable whose probability density function is a Dirac delta of mass one located at $-R$. Hence, we have that

$$\frac{1}{2\pi j} \int_{\nu - j\infty}^{\nu + j\infty} \frac{e^{\alpha R}}{2\alpha} \, \mathrm{d}\alpha = \frac{\mathbb{1}\{R > 0\}}{2}. \tag{51}$$

Similarly, the second one is a random variable whose probability density function is a Dirac delta of mass one located at and $-R + \log 2$ that evaluates as

$$\frac{1}{2\pi j} \int_{\nu - j\infty}^{\nu + j\infty} \frac{e^{\alpha R} 2^{-\alpha}}{2\alpha} \, \mathrm{d}\alpha = \frac{\mathbb{1}\{R > \log 2\}}{2}. \tag{52}$$

Defining $\delta(R)$ as

$$\delta(R) = \frac{1}{2\pi j} \int_{\nu - j\infty}^{\nu + j\infty} \theta_1(\alpha) \, \mathrm{d}\alpha, \tag{53}$$

within $0 < R < \log 2$ we have that

$$\delta(R) = \frac{1}{\log 2} - \frac{R}{\log 2} + \frac{1}{2}. \tag{54}$$

As a consequence, the expansion of the RCU is given by

$$\text{RCU}(n) = \frac{R}{\log 2} + \frac{1}{n}\left(\frac{1}{\log 2} - \frac{R}{\log 2} + \frac{1}{2}\right) + O\left(\frac{1}{n^2}\right). \tag{55}$$

The former expansion suggests that the error probability converges to the outage probability as $\frac{\delta(R)}{n}$, where $\delta(R)$ is a monotonically decreasing function of the rate.

V. Numerical Results

In this section, we compare the proposed error probability approximations with the Gallager bounds (3) and (7), and the simulated RCU (27). More specifically, we numerically evaluate the saddlepoint approximations (18), (32), and (40), as well as the expansion (55).

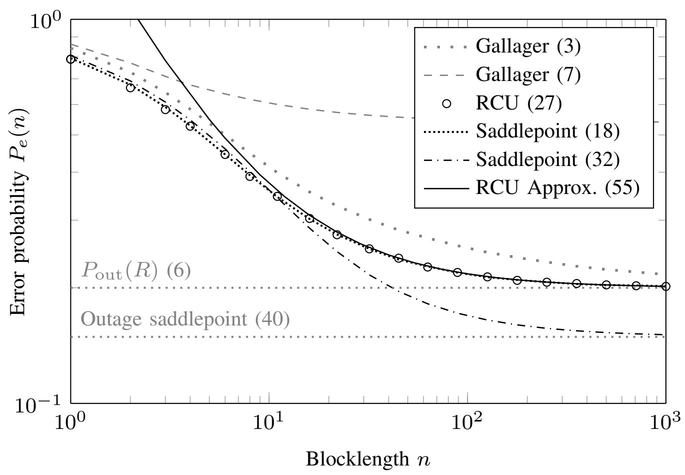

Fig. 2. Error probability bounds and approximations versus blocklength at $R = \frac{1}{5}\log 2$.

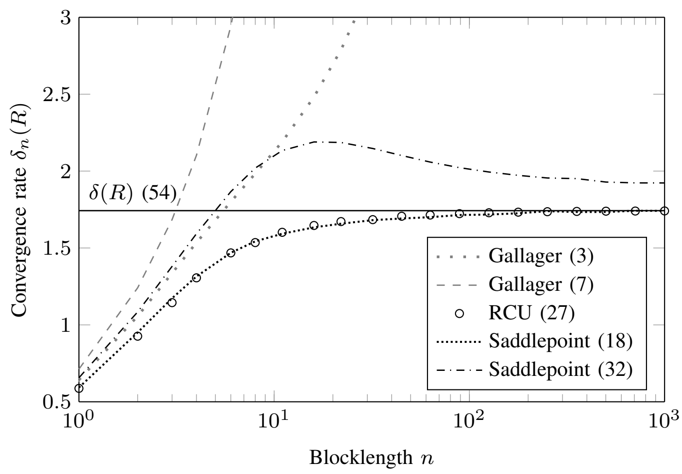

Fig. 3. Rate of convergence of the error probability to the outage probability versus codeword blocklength at $R = \frac{1}{5}\log 2$.

In Fig. 2, we observe that the saddlepoint approximation (18) is an accurate approximation of the RCU. As $n \to \infty$, the error probability converges to the outage probability (4), numerically confirming (24). Comparing the Gallager bound (3) with the saddlepoint approximation (18), and the Gallager bound (7) with the saddlepoint approximation (32), we note that in both cases the additive and the pre–exponential terms of the saddlepoint approximation provide a more refined characterization of the error probability. The contribution of these terms cannot be neglected in the quasi–static channel, since the exponential term of the error probability is not a dominant term when the error probability saturates.

A second observation from Fig. 2 is that, compared to the Gallager bound (7), the saddlepoint approximation (32) is tighter for small codeword blocklength. However, since the randomness of the erasure probability is considered in the approximation of the tail probability (27), this approximation exhibits a misadjustment for large blocklength, as it converges to the saddlepoint approximation of the outage probability (40), rather than to the actual outage probability (6).

Finally, we are interested in the convergence of the error probability to the outage probability. In particular, Fig. 3 depicts the convergence rate $\delta_n(R)$, defined as

$$\delta_n(R) = n\left(P_e(n) - P_{\mathrm{out}}(R)\right), \tag{56}$$

where $P_e(n)$ is a placeholder for the bounds and approximations of Fig. 3. Remarkably, Fig. 2 numerically illustrates that the Taylor expansion of the RCU (55) is a good approximation even for small codeword blocklength. Moreover, Fig. 3 illustrates that the error probability indeed exceeds the outage probability in a quantity that vanishes proportionally to $\frac{1}{n}$. That is,

$$\lim_{n\to\infty} \delta_n(R) = \delta(R). \tag{57}$$

As expected, none of the Gallager bounds provide the convergence in $\frac{1}{n}$, as the bounds are only tight for sufficiently large n. Contrarily, the saddlepoint approximation (32) does exhibit, although misadjusted, a convergence coefficient as $\frac{1}{n}$, whereas the saddlepoint approximation (18) leads to the correct convergence rate of the RCU (27).

VI. Conclusions

In this paper, we have derived refined approximations of the random–coding union bound in quasi–static binary erasure channels with uniformly distributed erasure probability. An expansion of the random–coding union bound in inverse powers of the codeword blocklength suggests that the error probability exceeds the outage probability by a quantity that is inversely proportional to the codeword blocklength.

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