Enhanced Belief Propagation Decoding of Polar Codes through Concatenation

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***Abstract*—The bit-channels of finite-length polar codes are not fully polarized, and a proportion of such bit-channels are neither completely "noiseless" nor completely "noisy". By using an outer low-density parity-check code for these intermediate channels, we show how the performance of belief propagation (BP) decoding of the overall concatenated polar code can be improved. A simple example reports an improvement in $\frac{E_b}{N_0}$ of 0.3 dB with respect to the conventional BP decoder.**

I. Introduction

Polar codes were proposed in [1] as a coding technique that provably achieves the capacity of symmetric binary-input discrete memoryless channels (B-DMCs) with low encoding and decoding complexity. The analysis and construction of polar codes are based on a successive cancellation (SC) decoder. Since then, decoders with better finite-length performance have been proposed. In [2], successive cancellation list (SCL) decoder was proposed and the performance was comparable to that of low-density parity-check (LDPC) codes. Belief propagation (BP) decoding of polar codes was proposed in [3], [4] with parallel and sequential message scheduling, respectively. Sequential BP was shown to perform better than parallel BP [4] over the polar code factor graph. BP decoding is also relevant in setups where soft-outputs need to be passed to other detectors in iterative processing structures, like in inter-symbol interference or multiple-antenna channels.

Concatenated polar codes with SC decoding have been considered. In particular, [5] reports near-exponential rate of decay of the error probability through a concatenation with outer Reed-Solomon codes. Instead, [6] proposes a concatenated code employing an outer polar code and inner block codes.

In this paper, we propose a concatenated polar coding scheme employing an inner polar code and an outer LDPC code for intermediate-quality bit-channels coupled with BP decoding. Fig. 1 shows the frame error rate (FER) and bit error rate (BER) for codes of length $N = 2^{12} = 4096$ and rate $R = \frac{1}{2}$ with different decoders. An instance of a concatenation with a Tanner code exhibits an improvement of over 0.3 dB over standard BP decoding of polar codes.

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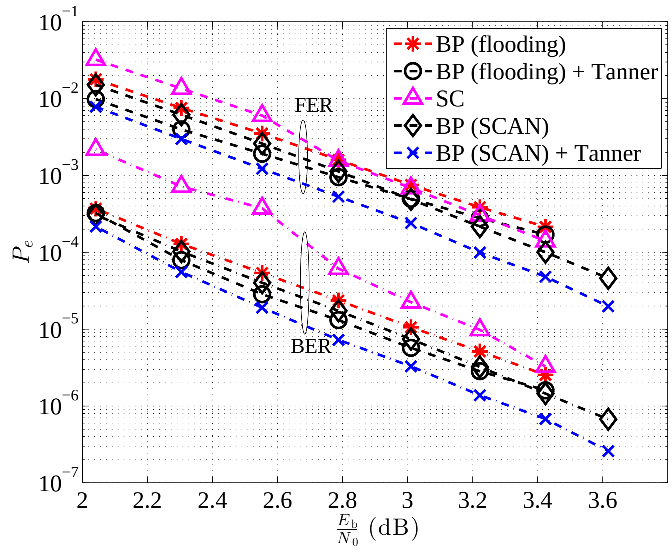

Fig. 1. Error rates of polar codes with $N = 4096$ and $R = \frac{1}{2}$ with various decoders over the AWGN channel. Bit-channels are sorted according to [7] at $\frac{E_b}{N_0} = 0$ dB.

II. Preliminaries

Throughout the paper, we define $[b] \stackrel{\text{def}}{=} \{1, \ldots, b\}$ for $b \in \mathbb{Z}$. We use x_1^b to denote a length-b vector $(x_1, \ldots, x_b)$ and $\boldsymbol{A}^\top$ denotes the transpose of matrix $\boldsymbol{A}$. Row vectors are assumed.

Let $W : \mathcal{X} \to \mathcal{Y}$ denote a B-DMC, with input alphabet $\mathcal{X} = \{0, 1\}$, output alphabet $\mathcal{Y}$, and transition probability $W(y|x), x \in \mathcal{X}, y \in \mathcal{Y}$. The channel mutual information with equiprobable inputs is denoted by $I(W)$ and the corresponding Bhattacharyya parameter by $Z(W)$. Let N be the block length, x_1^N, y_1^N be the channel input and output sequences, and the corresponding vector channel be $W^N\left(y_1^N|x_1^N\right)$.

A. Channel polarization

Consider the matrix $\boldsymbol{G}_2 = \left[\begin{smallmatrix}1 & 0\\ 1 & 1\end{smallmatrix}\right]$, and let $\boldsymbol{G}_N = \boldsymbol{G}_2^{\otimes n}$ be the $N \times N$ matrix corresponding to the Kronecker product of $\boldsymbol{G}_2$ with itself $n = \log_2 N$ times. Information bits are denoted by $u_1^N \in \{0, 1\}^N$. We define $W_N\left(y_1^N|u_1^N\right) = W^N\left(y_1^N|u_1^N \boldsymbol{G}_N\right)$ as the vector channel induced from the information bits. Then, out of $W^N\left(y_1^N|u_1^N\right)$, a SC decoder defines the channels

$$W_N^{(i)}\left(y_1^N, u_1^{i-1}|u_i\right) = \sum_{u_{i+1}^N} \frac{1}{2^{N-i}} W_N\left(y_1^N|u_1^N\right) \qquad (1)$$

for $i \in [N]$. The channel polarization theorem [1] states that $I\left(W_N^{(i)}\right)$ converges to either 0 or 1 as N tends to infinity.

Polar codes of rate $R = \frac{K}{N}$ are constructed by selecting the K indices $i \in [N]$ such that $I\left(W_N^{(i)}\right)$ is highest; the information bits corresponding to the remaining $N-K$ indices are *frozen* to zero. The set of frozen indices is the frozen set $\mathcal{F}$; its complement is denoted by $\mathcal{F}^c$. Equivalently, the generator matrix $\boldsymbol{G}_{\text{PC}}$ of a polar code is obtained by substituting the ith row in $\boldsymbol{G}_N$ with the all-zero vector for all $i \in \mathcal{F}$.

B. Successive cancellation decoding

In SC decoding, the bits corresponding to indices $i \in \mathcal{F}^c$ are estimated as

$$\widehat{U}_i = \arg\max_{u_i=0,1} W_N^{(i)}\left(y_1^N, u_1^{i-1}|u_i\right), i \in \mathcal{F}^c. \tag{2}$$

The decoding complexity of the SC decoder is $O(N \log N)$. It was shown in [8] that the probability of error under SC decoding decays as $o(2^{-N^\beta})$ for any fixed $\beta < \frac{1}{2}$.

C. Belief propagation decoding

BP decoding is a message passing decoding algorithm that has been extensively studied for decoding codes defined on graphs. BP decoding of polar codes has been considered in [3], [9], [10] and it was shown that the complexity of the BP decoding is $O(N \log N)$.

The factor graph of a polar code is a graphical representation of the generator matrix $\boldsymbol{G}_N$, interconnecting variable nodes (VNs) and check nodes (CNs). An instance for $N = 8$ is shown in Fig. 2.

Information bits $U_i, i \in [N]$, are represented as the set of leftmost VNs and are partitioned in 2 sets, $\mathcal{U}_{\mathcal{F}}$ and $\mathcal{U}_{\mathcal{F}^c}$, depending on whether the corresponding indices are in the frozen set or not. Obviously, $Z\left(W_N^{(i)}\right) \leqslant Z\left(W_N^{(j)}\right), \forall i \in \mathcal{F}^c, j \in \mathcal{F}$. Each column of VNs (CNs) is called a VN (CN) layer. The number of VN layers is $n+1$ and the number CN layers is n, where $n = \log_2 N$. In the sequel, we assume the layers are labeled from right to left as follows. Y_1^N (U_1^N) forms the 0th (nth) VN layer, and the CNs connected to Y_1^N (U_1^N) form the 1st (nth) CN layer. Note that in the ith layer, $i \in [n]$, there are $\frac{N}{2}$ Z-shaped VN-CN connections, one of which is highlighted in Fig. 2. The channel observation y_1^N, is fed to the rightmost VNs $Y_i, i \in [N]$ and the BP decoder passes messages along the graph in an iterative fashion according to the specific VN and CN update rules (see e.g. [11, Ch. 2]).

III. Enhanced BP with Concatenation

In this section, we introduce a concatenation scheme that improves the performance of polar codes under BP decoding. The Bhattacharyya parameters $Z\left(W_N^{(i)}\right), i \in [N]$ of finite-length polar codes no longer "fully" polarize, i.e., the proportion of bit channels for which $\delta < Z\left(W_N^{(i)}\right) < 1-\delta$ is not negligible, for small δ. For the example of Fig. 1 and $\delta = 0.01$, it is observed that $\frac{1}{N}\left|\{i : \delta < Z\left(W_N^{(i)}\right) < 1-\delta\}\right| \approx 0.22$. The top and bottom thresholds in Fig. 3 illustrate this proportion of channels.

Furthermore, the spread of the $Z\left(W_N^{(i)}\right), i \in [N]$ implies unequal protection of the corresponding information bits as well as small differences between the highest quality frozen channel and the lowest quality information channel.

A. Code construction and factor graph representation

For each channel $W_N^{(i)}, i \in [N]$, we have the following.

Definition 1. *Given $\delta_1, \delta_2 \in \mathbb{R}$ such that $0 < \delta_1 \leqslant \delta_2 < 1$, for all $i \in [N]$, we call a channel $W_N^{(i)}$* ***good*** *if $Z\left(W_N^{(i)}\right) < \delta_1$; we call a channel $W_N^{(i)}$* ***intermediate*** *if $\delta_1 \leqslant Z\left(W_N^{(i)}\right) < \delta_2$; and we call a channel* ***bad*** *if $Z\left(W_N^{(i)}\right) \geqslant \delta_2$.*

The main idea is the following. Uncoded data bits are transmitted through good channels; the input U_i to bad channels are frozen to 0; *coded* bits are transmitted through intermediate channels so that they are almost equally as well protected as the uncoded data bits on good channels.

Definition 2. *We denote a bipartite graph by $(\mathcal{V}, \mathcal{C}, \mathcal{E})$, where $\mathcal{V}$ is the set of VNs, $\mathcal{C}$ is the set of CNs, and $\mathcal{E}$ is the set of edges connecting $\mathcal{V}$ and $\mathcal{C}$.*

Let $(\mathcal{V}_{\text{std}}, \mathcal{C}_{\text{std}}, \mathcal{E}_{\text{std}})$ be the standard BP decoding graph of a polar code of length N. Let the set of VNs in the nth layer be partitioned into $\mathcal{U}_{\text{good}}, \mathcal{U}_{\text{inter}}$, and $\mathcal{U}_{\text{bad}}$, such that $Z\left(W_N^{(i)}\right) < \delta_1$, $\delta_1 \leqslant Z\left(W_N^{(i)}\right) < \delta_2$, and $Z\left(W_N^{(i)}\right) \geqslant \delta_2$, respectively, and let $\mathcal{F}_{\text{good}}, \mathcal{F}_{\text{inter}}$, and $\mathcal{F}_{\text{bad}}$ be the corresponding set of indices with $\mathcal{F}_{\text{good}} \cup \mathcal{F}_{\text{inter}} \cup \mathcal{F}_{\text{bad}} = [N]$.

Let $(\mathcal{V}_{\text{outer}}, \mathcal{C}_{\text{outer}}, \mathcal{E}_{\text{outer}})$ be a Tanner graph (bipartite graph) of an LDPC code of length $|\mathcal{V}_{\text{outer}}|$, rate R_{outer}, and normalized degree distribution (λ, ρ).

The enhanced BP decoding graph $(\mathcal{V}_{\text{ebp}}, \mathcal{C}_{\text{ebp}}, \mathcal{E}_{\text{ebp}})$ is formalized such that $\mathcal{V}_{\text{ebp}} = \mathcal{V}_{\text{std}}$, $\mathcal{C}_{\text{ebp}} = \mathcal{C}_{\text{std}} \cup \mathcal{C}_{\text{outer}}$, and $\mathcal{E}_{\text{ebp}} = \mathcal{E}_{\text{std}} \cup \mathcal{E}_{\text{outer}}$, where $\mathcal{V}_{\text{outer}} = \mathcal{U}_{\text{inter}}$.

According to the above definition, the rate of overall concatenated scheme is

$$R = \frac{\left|\mathcal{F}_{\text{good}}\right| + \left|\mathcal{F}_{\text{inter}}\right| \cdot R_{\text{outer}}}{N}, \tag{3}$$

$$= R_{\text{PC}} - \frac{\left|\mathcal{F}_{\text{inter}}\right| \cdot (1 - R_{\text{outer}})}{N}, \tag{4}$$

where

$$R_{\text{PC}} = \frac{\left|\mathcal{F}_{\text{good}}\right| + \left|\mathcal{F}_{\text{inter}}\right|}{N}. \tag{5}$$

is the rate of the inner polar code.

Fig. 2 shows an example of the enhanced BP decoding graph of a polar code of length $N = 8$. In layer $n = \log_2 N = 3$, $\mathcal{U}_{\text{bad}}$ consists of the top two VNs, $\mathcal{U}_{\text{inter}}$ consists of the middle five VNs, and $\mathcal{U}_{\text{good}}$ consists of the last VN.

B. Decoding

Scheduling, i.e., the order in which nodes generate their output messages, plays a key role in the performance and complexity of BP decoders [10]. There are two main types of scheduling to pass messages along the graph from Y_1^N to U_1^N and back.

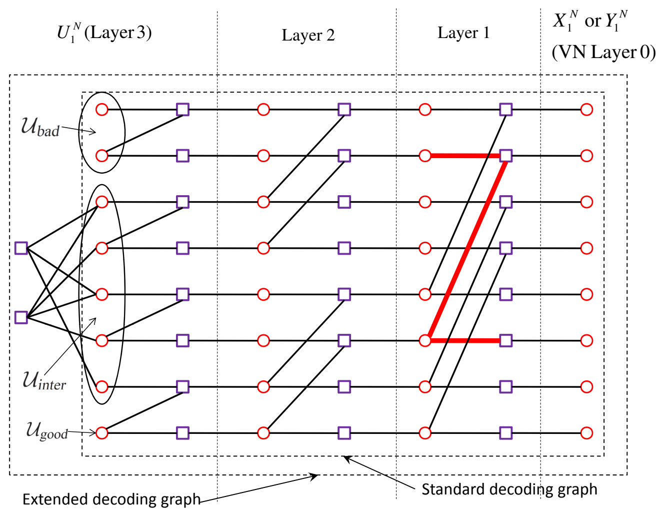

Fig. 2. Extended factor graph for of concatenated polar codes.

One is called "flooding," where messages are passed in parallel from the 0th layer (VNs corresponding to received codeword Y_1^N) to the nth layer (VNs corresponding to data bits U_1^N). Each layer consists of $\frac{N}{2}$ Z-shaped VN-CN connections (one such Z-shaped connection in the first layer is highlighted in Fig. 2). Upon receiving messages on U_1^N, the messages corresponding to VNs in $\mathcal{U}_{\text{inter}}$ are passed on as observations to the outer code decoder.

The other is called "SCAN" decoding [4], where the scheduling is similar to that of SC decoding. Note that SC decoding can be viewed as a special BP scheduling over the standard factor graph of polar codes [10], where the messages passed from Y_1^N to U_1^N are real-valued and the messages passed from U_1^N to Y_1^N are the 0-1 valued decisions on the data bits. The SCAN decoder has the same scheduling as the SC decoder, but instead of passing binary messages from U_1^N to Y_1^N, soft messages are passed. It is shown in [4] that SCAN decoders can improve the performance and reduce the complexity of the decoder.

1) Enhanced flooding BP: Let $L_{V\to C}(\lambda, i)$ and $L_{C\to V}(\lambda, i)$ be the messages from the VN on the ith row of the λth VN layer to the CN on the ith row of the $(\lambda+1)$th CN layer and the messages of the reverse direction, respectively, for $i \in [N], \lambda \in [0 : n-1]$. Let $L_{V\to C}(n, i)$ denote the messages flooded from the nth VN layer to $\mathcal{C}_{\text{outer}}$ as the channel input messages to the outer LDPC code, let $L_{C\to V}(n, i)$ denote the combined messages from CN in $\mathcal{C}_{\text{outer}}$ to VN on the ith row in the nth VN layer, which serve as the output messages from the outer code. Let $L_{\text{out}}^{\text{LDPC}}$=BP_Decoder_LDPC($L_{\text{in}}^{\text{LDPC}}, I$) be the BP decoding function of the LDPC code with observation messages $L_{\text{in}}^{\text{LDPC}}$, number of iterations I and output messages $L_{\text{out}}^{\text{LDPC}}$. Algorithm 1 describes the enhanced flooding BP decoder.

Algorithm 1. ENHANCED BP DECODER BY FLOODING

Input:
$L_{\text{in}} \in \mathbb{R}^N$: channel observation messages;
I_{max}: maximum number of iterations.
Output:
L_{out} : decoded output messages on VNs in the 0th layer.

For $i = 1$ **to** N
  $L_{V\to C}(0, i) \leftarrow L_{\text{in}}(i)$
end For
While number of iterations $< I_{\text{max}}$
  For $\lambda = 1$ **to** n
    For each Z-shaped connection in the λth layer
      Update CN-to-VN message on the lower edge
      Update VN-to-CN message on the diagonal edge
      Update CN-to-VN message on the upper edge
      Update CN-to-VN message on the diagonal edge
    end For
    For $i = 1$ **to** N
      Update $L_{V\to C}(\lambda, i)$
    end For
  end For
  $L_{C\to V}(n, i) \leftarrow$ BP_Decoder_LDPC($L_{\text{in}}^{\text{LDPC}}, 1$)
  For $\lambda = n-1$ **down to** 0
    For each Z-shaped connection in the λth layer
      Update VN-to-CN message on the upper edge
      Update CN-to-VN message on the diagonal edge
      Update VN-to-CN message on the lower edge
      Update VN-to-CN message on the diagonal edge
    end For
    For $i = 1$ **to** N
      Update $L_{C\to V}(\lambda, i)$
    end For
  end For
  For $i = 1$ **to** N
    If $(L_{C\to V}(0, i)) + L_{\text{in}(i)}) > 0$ **then**
      $\widehat{X}_i \leftarrow 0$
    Else
      $\widehat{X}_i \leftarrow 1$
    end If
  end For
  If Is_Parity_Check_Satisfied($\widehat{X}_1^N$) $= 1$
    $L_{\text{out}} \leftarrow L_{C\to V}(0, i) + L_{\text{in}}(i)$ and return
  end If
end While

It is not necessary to run all I_{max} iterations in Algorithm 1. After each iteration, if the estimated codeword $\widehat{X}_1^N$ satisfies the parity-check equation, the algorithm will stop and output the corresponding codeword. This early termination reduces the decoding complexity and is based on the observation that error events contain very few instances of decoding to a wrong codeword. Most errors happen when no codeword is found after running all I_{max} iterations. Algorithm 2 can be used to check for early termination.

Algorithm 2. Is_A_Codeword = Is_Parity_Check_Satisfied(X_1^N)

Input:
$X_1^N \in \{0, 1\}^N$: a row vector to check if it is a codeword.
Output:

Is_A_Codeword: equals 1 if X_1^N is a codeword of polar code, 0 otherwise.

$\widehat{U}_1^N \leftarrow X_1^N \cdot \boldsymbol{G}_{\text{PC}}$
$s \leftarrow \boldsymbol{H}_{\text{LDPC}} \cdot \left(\widehat{U}_{\mathcal{F}_{\text{inter}}}\right)^\top$
If $\widehat{U}_i = 0, \forall i \in \mathcal{U}_{\text{bad}}$ and $s = \mathbf{0}$
  Is_A_Codeword $\leftarrow 1$
Else
  Is_A_Codeword $\leftarrow 0$
end If

Since the parity-check matrix of a polar codes is not sparse, the complexity of checking whether $\widehat{X}_1^N$ is a codeword would be $O(N^2)$ if it is accomplished by checking each CN independently to verify if all parity-check equations are satisfied. However, according to the following proposition, Algorithm 2 has complexity $O(N \log N)$.

Proposition 1. *Suppose $\boldsymbol{G}_{PC}$ and $\boldsymbol{H}_{PC}$ are the generator and parity-check matrices of a polar code of length N and frozen set $\mathcal{F} \subset [N]$. A binary row vector $X_1^N \in \{0,1\}^N$ is a codeword, i.e., $\boldsymbol{H}_{PC} \cdot \left(X_1^N\right)^\top = \mathbf{0}$, if and only if*

$$\widehat{U}_i = 0, \forall i \in \mathcal{F}, \tag{6}$$

where $\widehat{U}_1^N = X_1^N \cdot \boldsymbol{G}_{PC}$.

2) Enhanced SCAN BP: We next give details of the SCAN BP decoder. In general, $L_{V\to C}(n,i), i \in [N]$, are obtained sequentially and thus sequential message-passing decoding of the outer code is used. Several sequential message-passing decoding schemes of LDPC decoding schemes that use partitioning of the CNs have been studied in [12], [13]. We describe the message update algorithm on edges connecting to a particular VN $V_i \in \mathcal{U}_{\text{inter}}$ in Algorithm 3.

Algorithm 3. Sequential message-passing of $V_i \in \mathcal{U}_{\text{inter}}$.

Input:
$L_{V\to C}(n,i)$: message from the ith VN in the nth layer, serving as the channel input message to the outer code.
Output:
$L_{C\to V}(n,i)$: combined message from $\mathcal{C}_{\text{outer}}$ to the ith VN in the nth layer.

For each edge $E_{i,j}$ connecting to V_i from CN $C_j \in \mathcal{C}_{\text{outer}}$
  Update message on $E_{i,j}$ from C_j to V_i
end For
$L_{C\to V}(n,i) \leftarrow \sum_j \{\text{message from } C_j \text{ to } V_i\}$
For each edge $E_{i,j}$ connecting to $C_j \in \mathcal{C}_{\text{outer}}$ from V_i
  Update message on $E_{i,j}$ from V_i to C_j
end For

The SCAN BP decoder of the concatenated code is completed by applying Algorithm 3 to VN V_i in nth layer whenever $L_{V\to C}(n,i)$ is obtained by SCAN. The updated $L_{C\to V}(n,i)$ is then fed back into the SCAN decoding of polar codes to update $L_{V\to C}(n,j)$ where $j > i$.

IV. Numerical Examples

In this section, we describe the simulation setup and the parameters corresponding to the performance results reported in Fig. 1. The length of the polar code is $N = 4096$, so there are $n = 12$ CN layers. The number of data bits is $K = 2048$ and thus the code rate is $R = \frac{1}{2}$.

A. Channel ordering

In our simulations, the channel qualities are measured by the corresponding Bhattacharyya parameters $Z\big(W_N^{(i)}\big), i \in [N]$ obtained using the algorithm in [7] at $\frac{E_b}{N_0} = 0$ dB, which is the Shannon limit for a rate-$\frac{1}{2}$ code with unconstrained inputs. The SNR region of interest (around 2.5 dB) is different from the region the polar code is optimized at (0 dB), which indicates that the all codes reported are designed for a mismatched SNR. From results not shown in the paper, our scheme performs universally better than standard polar codes and the improvement depends on the SNR mismatch between design SNR and real channel SNR.

From the definition of $Z\big(W_N^{(i)}\big), i \in [N]$, an SC decoder is implied. When BP decoding is used, SC decoding ordering is not necessarily the best possible. We set the thresholds $\delta_1 = 0.5736, \delta_2 = 0.83$ as illustrated in Fig. 3. Then $|\mathcal{F}_{\text{good}}| = |\mathcal{U}_{\text{good}}| = 1984$, $|\mathcal{F}_{\text{inter}}| = |\mathcal{U}_{\text{inter}}| = 155$, and $|\mathcal{F}_{\text{bad}}| = |\mathcal{U}_{\text{bad}}| = 1957$. Bits $U_i, i \in \mathcal{F}_{\text{bad}}$ are frozen to 0; uncoded data-bits are assigned to $U_i, i \in \mathcal{F}_{\text{good}}$; and LDPC coded data-bits are assigned to $U_i, i \in \mathcal{F}_{\text{inter}}$. For comparison, the threshold for a conventional rate-$\frac{1}{2}$ polar code is shown to be 0.70 in Fig. 3.

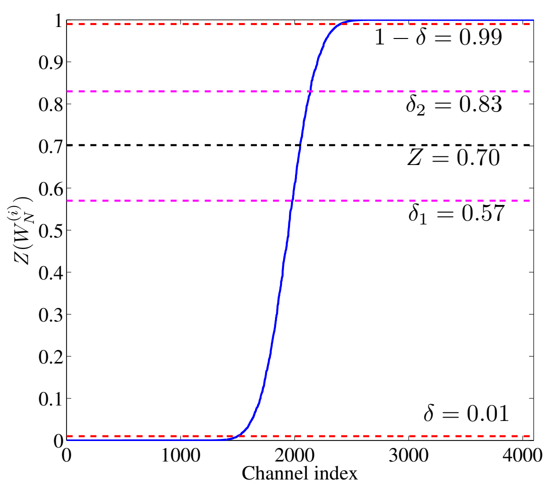

Fig. 3. Thresholds for sorted Bhattacharyya parameters $Z(W_N^{(i)})$.

B. Outer LDPC code

We use the $(3,5)$-regular Tanner code, with $|\mathcal{V}_{\text{outer}}| = 155$, $|\mathcal{C}_{\text{outer}}| = 93$, and $d_{\min} = 20$. Each VN is connected to 3 CNs and each CN is connected to 5 VNs. With these parameters, the rate of the overall concatenated code is $R = \frac{1}{2}$.

C. Results

Fig. 1 reports both BER and FER for the proposed polar code concatenation over the binary-input AWGN channel. Results are shown for both flooding and SCAN scheduling. For comparison purposes, results for three other decoding schemes are also shown, namely, conventional SC, BP, and SCAN.

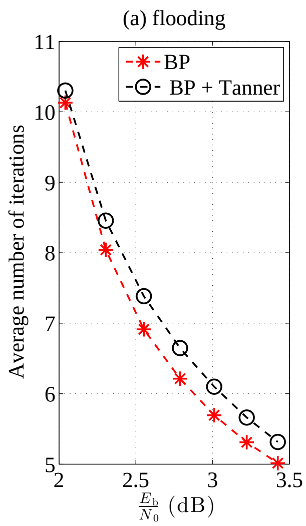

Fig. 4. Average number of iterations for flooding and SCAN BP.

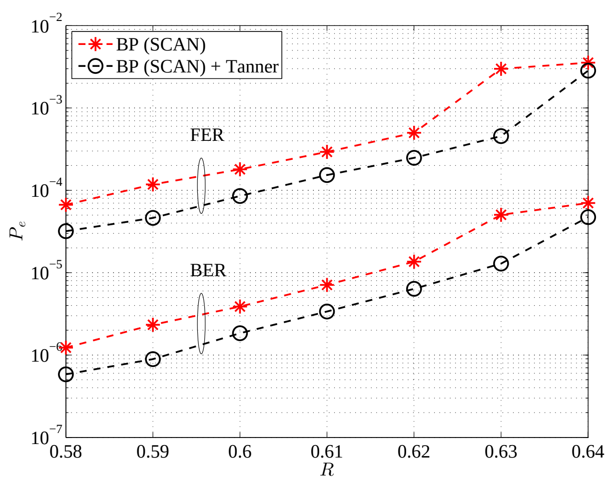

Fig. 5. BER/FER of BP SCAN over the AWGN channel at SNR = 4 dB.

Fig. 4 shows the average number of iterations needed to decode a codeword corresponding to the BP decoding schemes in Fig. 1. It is observed that the number of iterations needed by the concatenation with the Tanner code is decreased for SCAN, but increased for flooding.

Fig. 5 shows the performance of a polar code of length $N = 4096$ over the AWGN channel with SNR = 4 dB under standard SCAN BP decoding and SCAN BP decoding of the concatenated scheme with the Tanner code. The concatenated coding scheme with enhanced BP decoding offers an improvement in BER and FER over a range of code rates, and the absolute gain appears to be independent of the rate.

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