

# The Saddlepoint Approximation: Unified Random Coding Asymptotics for Fixed and Varying Rates

Jonathan Scarlett  
University of Cambridge  
jms265@cam.ac.uk

Alfonso Martinez  
Universitat Pompeu Fabra  
alfonso.martinez@ieee.org

Albert Guillén i Fàbregas  
ICREA & Universitat Pompeu Fabra  
University of Cambridge  
guillen@ieee.org

**Abstract**—This paper presents a saddlepoint approximation of the random-coding union bound of Polyanskiy *et al.* for i.i.d. random coding over discrete memoryless channels. The approximation is single-letter, and can thus be computed efficiently. Moreover, it is shown to be asymptotically tight for both fixed and varying rates, unifying existing achievability results in the regimes of error exponents, second-order coding rates, and moderate deviations. For fixed rates, novel exact-asymptotics expressions are specified to within a multiplicative  $1 + o(1)$  term. A numerical example is provided for which the approximation is remarkably accurate even at short block lengths.

## I. INTRODUCTION

In this paper, we consider problem of channel coding over a discrete memoryless channel  $W(y|x)$ . There exists extensive literature studying the tradeoff between the rate  $R$ , error probability  $p_e$  and block length  $n$ , including:

- 1) Error exponents ( $R < C$ , exponentially decaying  $p_e$ ) [1];
- 2) Second-order coding rates ( $R \rightarrow C$ , fixed  $p_e$ ) [2], [3];
- 3) Moderate deviations ( $R \rightarrow C$  and  $p_e \rightarrow 0$  simultaneously) [4],

where  $C$  is the capacity. These asymptotic notions provide valuable insight, but at finite block lengths it is generally unclear which one dictates the performance.

In [3, Sec. III], a non-asymptotic approach was taken. The most powerful of the achievability bounds therein is the random-coding union (RCU) bound, given by

$$\text{rcu}(n, M) \triangleq \mathbb{E} \left[ \min \left\{ 1, (M-1) \mathbb{P} \left[ W^n(\mathbf{Y}|\overline{\mathbf{X}}) \geq W^n(\mathbf{Y}|\mathbf{X}) \mid \mathbf{X}, \mathbf{Y} \right] \right\} \right], \quad (1)$$

where  $M = e^{nR}$  is the number of messages,  $(\mathbf{X}, \mathbf{Y}, \overline{\mathbf{X}}) \sim Q^n(\mathbf{x})W^n(\mathbf{y}|\mathbf{x})Q^n(\overline{\mathbf{x}})$ ,  $W^n(\mathbf{y}|\mathbf{x}) \triangleq \prod_{i=1}^n W(y_i|x_i)$ , and  $Q^n(\mathbf{x}) \triangleq \prod_{i=1}^n Q(x_i)$  for some input distribution  $Q$  (here we focus on i.i.d. random coding). The RCU bound has been shown to be close to non-asymptotic converse bounds in several numerical examples [3], but its computation is generally prohibitively complex beyond symmetric setups.

In [5], a saddlepoint approximation [6] was derived for a weakened bound, obtained from (1) using Markov's inequality:

$$\text{rcu}_s(n, M) \triangleq \mathbb{E} \left[ \min \left\{ 1, (M-1) e^{-i_s^n(\mathbf{X}, \mathbf{Y})} \right\} \right], \quad (2)$$

This work has been funded in part by the European Research Council under ERC grant agreement 259663, by the European Union's 7th Framework Programme (PEOPLE-2011-CIG) under grant agreement 303633 and by the Spanish Ministry of Economy and Competitiveness under grants RYC-2011-08150 and TEC2012-38800-C03-03.

where  $s > 0$  is arbitrary, and we define the generalized information density

$$i_s^n(\mathbf{x}, \mathbf{y}) \triangleq \sum_{i=1}^n i_s(x_i, y_i) \quad (3)$$

$$i_s(x, y) \triangleq \log \frac{W(y|x)^s}{\sum_{\overline{x}} Q(\overline{x})W(y|\overline{x})^s}. \quad (4)$$

The approximation in [5] is *single-letter* and takes the form  $\widehat{\text{rcu}}_s(n, M) = \alpha_n(Q, R, s) e^{-nE_r(Q, R, s)}$ , where  $E_r$  and  $\alpha_n$  represent the error exponent and the subexponential prefactor respectively. Numerical examples in [5] showed the approximation to be remarkably tight, while being essentially as easy to compute as the exponent alone. However, its derivation used heuristic arguments. The techniques of this paper formalize these arguments, and yield

$$\lim_{n \rightarrow \infty} \frac{\widehat{\text{rcu}}_s(n, M_n)}{\text{rcu}_s(n, M_n)} = 1 \quad (5)$$

at both fixed and varying rates. Moreover, both the lattice and non-lattice case (see Section III) are handled. Since  $\text{rcu}_s$  can be used to derive the random-coding exponent [1, Ch. 5], channel dispersion [3] and moderate deviations result [4], we conclude from (5) that  $\widehat{\text{rcu}}_s$  unifies these regimes.

In Theorem 1 below, we present a refined asymptotic bound  $\text{rcu}_s^*$  and a corresponding saddlepoint approximation  $\widehat{\text{rcu}}_s^*$  which is tight in the sense of (5), and which is seen to approximate the more powerful bound  $\text{rcu}$  remarkably well numerically (see Figure 1). This saddlepoint approximation not only unifies the above-mentioned regimes, but also characterizes the higher-order asymptotics. In particular, for a fixed error probability the approximation captures the third-order  $\frac{1}{2} \log n$  term [7, Sec. 3.4.5], and for a fixed rate we obtain the prefactor growth rate derived in [8] (see also [9]), along with a novel characterization of the multiplicative  $O(1)$  terms.

## II. PRELIMINARY DEFINITIONS AND RESULTS

We henceforth make use of the standard asymptotic notations  $O(\cdot)$ ,  $o(\cdot)$ ,  $\Theta(\cdot)$ ,  $\Omega(\cdot)$  and  $\omega(\cdot)$ .

1) *Information Density Moments and  $E_0$  Function*: We write the mean and variance of the information density as

$$I_s(Q) \triangleq \mathbb{E}[i_s(X, Y)] \quad (6)$$

$$U_s(Q) \triangleq \text{Var}[i_s(X, Y)], \quad (7)$$

where  $(X, Y) \sim Q \times W$ . Note that  $I_1(Q) = I(X; Y)$ .

Following Gallager [1, Ch. 5], we define the  $E_0$  function

$$E_0(Q, \rho, s) \triangleq -\log \mathbb{E}[e^{-\rho i_s(X, Y)}] \quad (8)$$

and the random-coding error exponent

$$E_r(Q, R) \triangleq \sup_{s>0, \rho \in [0, 1]} E_0(Q, \rho, s) - \rho R. \quad (9)$$

While the supremum is achieved by  $s = \frac{1}{1+\rho}$  [1, Ex. 5.6], it will be convenient to consider an arbitrary choice of  $s > 0$ .

The optimal  $\rho$  in (9) for a given value of  $s$  is denoted by

$$\hat{\rho}(Q, R, s) \triangleq \arg \max_{\rho \in [0, 1]} E_0(Q, \rho, s) - \rho R, \quad (10)$$

and the critical rate is defined as

$$R_s^{\text{cr}}(Q) \triangleq \sup \{R : \hat{\rho}(Q, R, s) = 1\}. \quad (11)$$

We define the following derivatives associated with (10):

$$c_1(Q, R, s) \triangleq R - \left. \frac{\partial E_0(Q, \rho, s)}{\partial \rho} \right|_{\rho=\hat{\rho}(Q, R, s)} \quad (12)$$

$$c_2(Q, R, s) \triangleq - \left. \frac{\partial^2 E_0(Q, \rho, s)}{\partial \rho^2} \right|_{\rho=\hat{\rho}(Q, R, s)}. \quad (13)$$

The following properties of the above quantities coincide with those given by Gallager [1, pp. 141-143], and follow by adapting the arguments therein to the case of a fixed  $s > 0$ :

- If  $U_s(Q) > 0$ , then  $c_2 > 0$  for all  $R$ ;
- For  $R \in [0, R_s^{\text{cr}}(Q)]$ , we have  $\hat{\rho} = 1$  and  $c_1 < 0$ ;
- For  $R \in [R_s^{\text{cr}}(Q), I_s(Q)]$ ,  $\hat{\rho}$  is strictly decreasing in  $R$ , and  $c_1 = 0$ ;
- For  $R > I_s(Q)$ , we have  $\hat{\rho} = 0$  and  $c_1 > 0$ .

Here and throughout the paper, the arguments to  $\hat{\rho}$ ,  $c_1$ , etc. are omitted when their values are clear from the context.

2) *Singular vs. Non-Singular Case:* Given an input distribution  $Q$  and channel  $W$ , we define the set

$$\mathcal{Y}_1(Q) \triangleq \left\{ y : W(y|x) \neq W(y|\bar{x}) \text{ for some } x, \bar{x} \right. \\ \left. \text{such that } Q(x)Q(\bar{x})W(y|x)W(y|\bar{x}) > 0 \right\}. \quad (14)$$

Following the terminology of Altuğ and Wagner [8], we say that  $(Q, W)$  is singular if  $\mathcal{Y}_1(Q) = \emptyset$ , and non-singular otherwise. Our techniques can be used to handle both cases. In the singular case, we in fact have  $\text{rcu} = \text{rcu}_s$  [10], and hence (5) gives the desired result regarding the approximation of  $\text{rcu}$ . In fact, our analysis can be applied directly to the dependence-testing (DT) bound [3], which improves (slightly) on  $\text{rcu}$  for singular channels. We focus on the non-singular case, and refer the reader to [10] for the singular case.

3) *Further Definitions:* We say that  $Z$  is a lattice random variable with offset  $\gamma$  and span  $h$  if its support is a subset of the lattice  $\{\gamma + ih : i \in \mathbb{Z}\}$ , and the same cannot remain true by increasing  $h$ . Our main result treats two cases separately depending on whether  $i_s(X, Y)$  is a lattice variable.

The density of a  $N(\mu, \sigma^2)$  random variable is denoted by

$$\phi(z; \mu, \sigma^2) \triangleq \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(z-\mu)^2}{2\sigma^2}}. \quad (15)$$

In the lattice case, we similarly write

$$\phi_h(z; \mu, \sigma^2) \triangleq \frac{h}{\sqrt{2\pi\sigma^2}} e^{-\frac{(z-\mu)^2}{2\sigma^2}}. \quad (16)$$

The remaining definitions are somewhat more technical. We define the reverse conditional distribution

$$\tilde{P}_s(x|y) \triangleq \frac{Q(x)W(y|x)^s}{\sum_{\bar{x}} Q(\bar{x})W(y|\bar{x})^s}, \quad (17)$$

the joint tilted distribution

$$P_{\hat{\rho}, s}^*(x, y) = \frac{Q(x)W(y|x)e^{-\hat{\rho}i_s(x, y)}}{\sum_{x', y'} Q(x')W(y'|x')e^{-\hat{\rho}i_s(x', y')}}, \quad (18)$$

and the conditional variance

$$c_3(Q, R, s) \triangleq \mathbb{E} \left[ \text{Var} [i_s(X_s^*, Y_s^*) | Y_s^*] \right], \quad (19)$$

where  $(X_s^*, Y_s^*) \sim P_{\hat{\rho}, s}^*(y)\tilde{P}_s(x|y)$ , and  $P_{\hat{\rho}, s}^*(y)$  is the  $y$ -marginal of (18). We have [9, Eq. (61)]

$$\text{Var}_{\tilde{P}_s(\cdot|y)} [i_s(X_s, y)] > 0 \iff y \in \mathcal{Y}_1(Q). \quad (20)$$

Furthermore, using (18), we have  $P_{\hat{\rho}, s}^*(y) > 0$  if and only if  $\sum_x Q(x)W(y|x) > 0$ . Combining these, we see that the non-singularity assumption implies  $c_3 > 0$  for all  $R$  and  $s > 0$ .

Finally, we define

$$\mathcal{I}_s \triangleq \left\{ i_s(x, y) : Q(x)W(y|x) > 0, y \in \mathcal{Y}_1(Q) \right\} \quad (21)$$

$$\psi_s \triangleq \begin{cases} 1 & \mathcal{I}_s \text{ does not lie on a lattice} \\ \frac{\bar{h}}{1-e^{-\bar{h}}} & \mathcal{I}_s \text{ lies on a lattice with span } \bar{h}. \end{cases} \quad (22)$$

### III. MAIN RESULT

Our saddlepoint approximation is written in the form

$$\widehat{\text{rcu}}_s^*(n, M) \triangleq \beta_n(Q, R, s) e^{-n(E_0(Q, \hat{\rho}, s) - \hat{\rho}R)}. \quad (23)$$

We treat the lattice and non-lattice cases separately, defining

$$\beta_n \triangleq \begin{cases} \beta_n^{\text{nl}} & i_s(X, Y) \text{ is non-lattice} \\ \beta_n^{\text{l}} & R - i_s(X, Y) \text{ has offset } \gamma \text{ and span } h, \end{cases} \quad (24)$$

where

$$\beta_n^{\text{nl}}(Q, R, s) \triangleq \int_{\log \frac{\sqrt{2\pi n c_3}}{\psi_s}}^{\infty} e^{-\hat{\rho}z} \phi(z; n c_1, n c_2) dz \\ + \frac{\psi_s}{\sqrt{2\pi n c_3}} \int_{-\infty}^{\log \frac{\sqrt{2\pi n c_3}}{\psi_s}} e^{(1-\hat{\rho})z} \phi(z; n c_1, n c_2) dz, \quad (25)$$

$$\beta_n^{\text{l}}(Q, R, s) \triangleq \sum_{i=i^*}^{\infty} e^{-\hat{\rho}(\gamma_n + ih)} \phi_h(\gamma_n + ih; n c_1, n c_2) \\ + \frac{\psi_s}{\sqrt{2\pi n c_3}} \sum_{i=i^*-1}^{i^*} e^{(1-\hat{\rho})(\gamma_n + ih)} \phi_h(\gamma_n + ih; n c_1, n c_2), \quad (26)$$

and where in (26) we define

$$\gamma_n \triangleq \min \left\{ n\gamma + ih : i \in \mathbb{Z}, n\gamma + ih \geq 0 \right\}, \quad (27)$$

$$i^* \triangleq \min \left\{ i \in \mathbb{Z} : \gamma_n + ih \geq \log \frac{\sqrt{2\pi n c_3}}{\psi_s} \right\}. \quad (28)$$

While (25) and (26) are written in terms of integrals and summations, both are single-letter and can be computed efficiently, with a complexity which is independent of  $n$ . In the non-lattice case, this is done by noting that

$$\int_a^\infty e^{bz} \phi(z; \mu, \sigma^2) dz = e^{\mu b + \frac{1}{2} \sigma^2 b^2} Q\left(\frac{a - \mu - b\sigma^2}{\sigma}\right). \quad (29)$$

In the lattice case, we can write each summation in (26) as

$$\sum_i e^{b_0 + b_1 i + b_2 i^2} = e^{-\frac{b_1^2}{4b_2} + b_0} \sum_i e^{b_2 (i + \frac{b_1}{2b_2})^2}, \quad (30)$$

where  $b_2 < 0$ . We can thus obtain an accurate approximation by keeping only the terms in the summation such that  $i$  is sufficiently close to  $-\frac{b_1}{2b_2}$ . Overall, the computational complexity for any given  $s > 0$  is similar to that of computing the error exponent alone. In principle, the parameter  $s$  may be further optimized, but numerical studies indicate that it suffices to choose  $s = \frac{1}{1+\hat{\rho}}$  (i.e. the value maximizing  $E_0(Q, \hat{\rho}, s)$ ).

**Theorem 1.** Fix the input distribution  $Q$ , constant  $s > 0$ , and sequence of positive integers  $\{M_n\}_{n \geq 1}$ . If the pair  $(Q, W)$  is non-singular, then

$$\text{rcu}(n, M_n) \leq \text{rcu}_s^*(n, M_n)(1 + o(1)), \quad (31)$$

where

$$\text{rcu}_s^*(n, M) \triangleq \mathbb{E} \left[ \min \left\{ 1, \frac{M \psi_s}{\sqrt{2\pi n c_3}} e^{-i_s^n(\mathbf{X}, \mathbf{Y})} \right\} \right]. \quad (32)$$

Furthermore, we have

$$\lim_{n \rightarrow \infty} \frac{\widehat{\text{rcu}}_s^*(n, M_n)}{\text{rcu}_s^*(n, M_n)} = 1. \quad (33)$$

*Proof:* See Section IV-B. ■

The proof of Theorem 1 reveals that for a fixed target error probability we have  $\widehat{\text{rcu}}_s^* = \text{rcu}_s^* + O(\frac{1}{\sqrt{n}})$ . From the analysis given in [7, Sec. 3.4.5], setting  $\text{rcu}_s^* = \epsilon$  and solving for the required number of messages yields

$$\log M = nI_s(Q) - \sqrt{nU_s(Q)}Q^{-1}(\epsilon) + \frac{1}{2} \log n + O(1). \quad (34)$$

By Taylor expanding the  $Q^{-1}$  function, we conclude that the same is true of  $\widehat{\text{rcu}}_s^*$ . Note that since  $I_1(Q) = I(X; Y)$ , (34) is primarily of interest when  $s = 1$  and  $Q$  achieves capacity.

For a fixed rate  $R \geq 0$ , we can apply asymptotic expansions to (25)–(26) to show the following [10] (here  $f_n \asymp g_n$  means that  $\lim_{n \rightarrow \infty} \frac{f_n}{g_n} = 1$ ):

- If  $R \in [0, R_s^{\text{cr}}(Q))$ , then  $\beta_n(Q, R, s) \asymp \frac{\psi_s}{\sqrt{2\pi n c_3}}$ .
- If  $R = R_s^{\text{cr}}(Q)$ , then  $\beta_n(Q, R, s) \asymp \frac{\psi_s}{2\sqrt{2\pi n c_3}}$ .
- If  $R \in (R_s^{\text{cr}}(Q), I_s(Q))$ , then

$$\beta_n^{\text{nl}}(Q, R, s) \asymp \left( \frac{\psi_s}{\sqrt{2\pi n c_3}} \right)^{\hat{\rho}} \frac{1}{\sqrt{2\pi n c_2 \hat{\rho} (1 - \hat{\rho})}}, \quad (35)$$

$$\begin{aligned} \beta_n^1(Q, R, s) &\asymp \left( \frac{\psi_s}{\sqrt{2\pi n c_3}} \right)^{\hat{\rho}} \frac{h}{\sqrt{2\pi n c_2}} \\ &\times \left( e^{-\hat{\rho} \gamma'_n} \left( \frac{1}{1 - e^{-\hat{\rho} h}} \right) + e^{(1-\hat{\rho}) \gamma'_n} \left( \frac{e^{-(1-\hat{\rho}) h}}{1 - e^{-(1-\hat{\rho}) h}} \right) \right), \end{aligned} \quad (36)$$

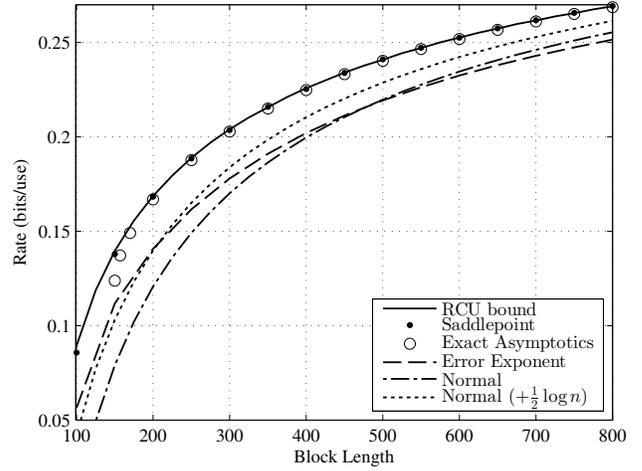


Figure 1. Rate required to achieve a target error probability  $\epsilon = 10^{-5}$  for the binary symmetric channel with crossover probability  $\delta = 0.15$ , and the uniform input distribution  $Q = (\frac{1}{2}, \frac{1}{2})$ . This corresponds to the lattice case in (24). The capacity and critical rate are 0.390 bits/use and 0.124 bits/use.

where  $\gamma'_n \triangleq \gamma_n + i^* h - \log \frac{\sqrt{2\pi n c_3}}{\psi_s} \in [0, h)$  (see (28)).

- If  $R = I_s(Q)$ , then  $\beta_n(Q, R, s) \asymp \frac{1}{2}$ .
- If  $R > I_s(Q)$ , then  $\beta_n(Q, R, s) \asymp 1$ .

When combined with Theorem 1, these expansions provide an alternative proof of the main result of Altuğ-Wagner [8], and an explicit characterization of the multiplicative  $O(1)$  terms.

#### A. Numerical Example

A numerical example is given in Figure 1 (see the caption for details). Definitions of the error exponent and normal approximations can be found in [3], and the exact asymptotics approximation equals the right-hand side of (36). We set  $s = 1$  for the normal approximation, and  $s = \frac{1}{1+\hat{\rho}}$  for the other approximations.

We see that the saddlepoint approximation provides an excellent approximation of  $\text{rcu}(n, M)$ . The exact asymptotics approximation is accurate other than a divergence near the critical rate. A similar divergence also occurs near capacity, but this is not visible in the plot; see [10] for further discussion. In this example, neither the error exponent approximation nor normal approximation is accurate, though the latter moves closer to  $\text{rcu}$  upon including the  $\frac{1}{2} \log n$  term. Roughly speaking, the normal (respectively, error exponent) approximation is better suited to rates near capacity (respectively, low rates), whereas the saddlepoint approximation is accurate at all rates.

It should be noted that the observed accuracy of the saddlepoint approximation is not limited to symmetric setups; see [5], [10] for further examples.

## IV. PROOF OF THEOREM 1

Due to space constraints, we omit some details and focus on the non-lattice case. Full details can be found in [10].

#### A. Proof of (33)

1) *Alternative Expressions for  $\text{rcu}_s^*$ :* For any non-negative random variable  $A$ , we have  $\mathbb{E}[\min\{1, A\}] = \mathbb{P}[A \geq U]$ ,

where  $U$  is uniform on  $(0, 1)$  and independent of  $A$ . Defining  $g_n \triangleq \frac{1}{\psi_s} \sqrt{2\pi n c_3}$ , we can thus write (32) as

$$\text{rcu}_s^*(n, M) = \mathbb{P}\left[nR - \sum_{i=1}^n i_s(X_i, Y_i) \geq \log(Ug_n)\right]. \quad (37)$$

Let  $F(t)$  denote the cumulative distribution function (CDF) of  $R - i_s(X, Y)$  and let  $Z_1, \dots, Z_n$  be i.i.d. with CDF

$$F_Z(z) = e^{E_0 - \hat{\rho}R} \int_{-\infty}^z e^{\hat{\rho}t} dF(t), \quad (38)$$

where the arguments to  $E_0$  are kept implicit. Using a standard change of measure argument, we showed in [9, Eq. (44)] that

$$\text{rcu}_s^*(n, M) = I_n e^{-n(E_0(Q, \hat{\rho}, s) - \hat{\rho}R)}, \quad (39)$$

where

$$I_n \triangleq \int_0^1 \int_{\log(ug_n)}^{\infty} e^{-\hat{\rho}z} dF_n(z) dF_U(u), \quad (40)$$

and where  $F_n$  is the CDF of  $\sum_{i=1}^n Z_i$ , and  $F_U$  is the CDF of  $U$ . Moreover, we showed in [9, Eqs. (48)–(49)] that

$$\mathbb{E}[Z] = c_1, \quad \text{Var}[Z] = c_2, \quad (41)$$

where  $c_1$  and  $c_2$  are defined in (12)–(13). It is not difficult to show that the non-singularity assumption implies  $U_s(Q) > 0$ , which in turn implies  $c_2 > 0$  (see Section II).

Since the integrand in (40) is non-negative, we can safely interchange the order of integration, yielding

$$I_n = \int_{-\infty}^{\infty} \int_0^{\min\{1, \frac{1}{g_n} e^z\}} e^{-\hat{\rho}z} dF_U(u) dF_n(z) \quad (42)$$

$$= \int_{\log g_n}^{\infty} e^{-\hat{\rho}z} dF_n(z) + \frac{1}{g_n} \int_{-\infty}^{\log g_n} e^{(1-\hat{\rho})z} dF_n(z), \quad (43)$$

where (43) follows by splitting the integral according to which value achieves the  $\min\{\cdot, \cdot\}$  in (42). Letting  $\hat{F}_n$  denote the CDF of  $\frac{\sum_{i=1}^n Z_i - nc_1}{\sqrt{nc_2}}$ , we can write (43) as

$$I_n = \int_{\frac{\log g_n - nc_1}{\sqrt{nc_2}}}^{\infty} e^{-\hat{\rho}(z\sqrt{nc_2} + nc_1)} d\hat{F}_n(z) + \frac{1}{g_n} \int_{-\infty}^{\frac{\log g_n - nc_1}{\sqrt{nc_2}}} e^{(1-\hat{\rho})(z\sqrt{nc_2} + nc_1)} d\hat{F}_n(z). \quad (44)$$

2) *Application of a Refined Central Limit Theorem:* Let  $\Phi(z)$  denote the CDF of a zero-mean unit-variance Gaussian random variable. Using the fact that  $\mathbb{E}[Z] = c_1$  and  $\text{Var}[Z] = c_2 > 0$ , we have from the refined central limit theorem in [11, Sec. XVI.4, Thm. 1] that

$$\hat{F}_n(z) = \Phi(z) + G_n(z) + \tilde{F}_n(z), \quad (45)$$

where  $\tilde{F}_n(z) = o(n^{-\frac{1}{2}})$  uniformly in  $z$ , and

$$G_n(z) \triangleq \frac{K}{\sqrt{n}} (1 - z^2) e^{-\frac{1}{2}z^2} \quad (46)$$

for some constant  $K$  depending only on the variance and third absolute moment of  $Z$ . Substituting (45) into (44), we obtain

$$I_n = I_{1,n} + I_{2,n} + I_{3,n}, \quad (47)$$

where the three terms denote the right-hand side of (44) with  $\Phi$ ,  $G_n$  and  $\tilde{F}_n$  respectively in place of  $\hat{F}_n$ . By reversing the step from (43) to (44), we see that  $I_{1,n}$  is precisely  $\beta_n^{\text{nl}}$  in (25). In accordance with the theorem statement, we must show that  $I_{2,n} = o(\beta_n^{\text{nl}})$  and  $I_{3,n} = o(\beta_n^{\text{nl}})$  even when  $R$  and  $\hat{\rho}$  vary with  $n$ . Let  $R_n \triangleq \frac{1}{n} \log M_n$  and  $\hat{\rho}_n \triangleq \hat{\rho}(Q, R_n, s)$ , and let  $c_{1,n}$  and  $c_{2,n}$  be the corresponding values of  $c_1$  and  $c_2$ . We assume with no real loss of generality that

$$\lim_{n \rightarrow \infty} R_n = R^* \quad (48)$$

for some  $R^* \geq 0$  possibly equal to  $\infty$ . Once (33) is proved for all such  $R^*$ , the same will follow for arbitrary  $\{R_n\}$ .

Table I summarizes the growth rates  $\beta_n^{\text{nl}}$ ,  $I_{2,n}$  and  $I_{3,n}$  for various ranges of  $R^*$ , and indicates whether the first or second integral (see (44)) dominates the behavior of each. We see that  $I_{2,n} = o(\beta_n^{\text{nl}})$  and  $I_{3,n} = o(\beta_n^{\text{nl}})$  for all  $R^*$ , as desired.

As an example, we consider the case  $R^* \in (R_s^{\text{cr}}(Q), I_s(Q))$ . The given behavior of  $\beta_n^{\text{nl}}$  follows immediately from (35). Taking the derivative of  $G_n(z)$  in (46), we can evaluate  $I_{2,n}$  by writing it in terms of the standard Gaussian density  $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$ . For  $I_{3,n}$ , we analyze the two integrals in a similar fashion; here we focus on the first. For the integration range given, the integrand is upper bounded by  $e^{-\hat{\rho} \log g_n} = \Theta(n^{-\hat{\rho}/2})$ . Combining this with the fact that  $\tilde{F}_n(z) = o(n^{-\frac{1}{2}})$  uniformly in  $z$ , we obtain the desired  $o(n^{-\frac{1}{2}(1+\hat{\rho})})$  decay rate.

## B. Proof of (31)

To prove (31), we make use of two technical lemmas, whose proofs can be found in [10, Appendix F].

**Lemma 1.** Fix  $K > 0$ , and for each  $n$ , let  $(n_1, \dots, n_K)$  be integers such that  $\sum_k n_k = n$ . Fix the probability mass functions (PMFs)  $Q_1, \dots, Q_K$  on a common finite alphabet, and let  $\sigma_1^2, \dots, \sigma_K^2$  be the corresponding variances. Let  $Z_1, \dots, Z_n$  be independent random variables,  $n_k$  of which are distributed according to  $Q_k$  for each  $k$ . Suppose that  $\min_k \sigma_k > 0$  and  $\min_k n_k = \Theta(n)$ . Defining

$$\mathcal{I}_0 \triangleq \bigcup_{k: \sigma_k > 0} \{z : Q_k(z) > 0\} \quad (49)$$

$$\psi_0 \triangleq \begin{cases} 1 & \mathcal{I}_0 \text{ does not lie on a lattice} \\ \frac{h_0}{1 - e^{-h_0}} & \mathcal{I}_0 \text{ lies on a lattice with span } h_0, \end{cases} \quad (50)$$

the sum  $S_n \triangleq \sum_i Z_i$  satisfies the following uniformly in  $t$ :

$$\mathbb{E}\left[e^{-S_n} \mathbb{1}\{S_n \geq t\}\right] \leq e^{-t} \left( \frac{\psi_0}{\sqrt{2\pi V_n}} + o\left(\frac{1}{\sqrt{n}}\right) \right), \quad (51)$$

where  $V_n \triangleq \text{Var}[S_n]$ , and  $\mathbb{1}\{\cdot\}$  is the indicator function.

*Proof:* The proof is analogous to that of [3, Lemma 47], except that the use of the Berry-Esseen theorem is replaced by the local limit theorems in [12, Thm. 1] and [13, Sec. VII.1, Thm. 2] for the non-lattice and lattice cases respectively. ■

Define the random variables

$$(\mathbf{X}, \mathbf{Y}, \bar{\mathbf{X}}, \mathbf{X}_s) \sim Q^n(\mathbf{x}) W^n(\mathbf{y}|\mathbf{x}) Q^n(\bar{\mathbf{x}}) \tilde{F}_s^n(\mathbf{x}_s|\mathbf{y}), \quad (52)$$

Table I  
GROWTH RATES OF  $\beta_n^{\text{nl}}$ ,  $I_{2,n}$  AND  $I_{3,n}$  WHEN THE RATE CONVERGES TO  $R^*$ .

	$\hat{\rho}$	$c_1$	Dominant Term(s)	$\beta_n^{\text{nl}}$	$I_{2,n}$	$I_{3,n}$
$R^* \in [0, R_s^{\text{cr}}(Q))$	1	$< 0$	2	$\Theta\left(\frac{1}{\sqrt{n}}\right)$	$\Theta\left(\frac{1}{n}\right)$	$o\left(\frac{1}{n}\right)$
$R^* = R_s^{\text{cr}}(Q)$	$\rightarrow 1$	$\rightarrow 0$	2	$\omega\left(\frac{1}{n}\right)$	$O\left(\frac{1}{n}\right)$	$o\left(\frac{1}{n}\right)$
$R^* \in (R_s^{\text{cr}}(Q), I_s(Q))$	$\in (0, 1)$	0	1,2	$\Theta\left(\frac{1}{n^{\frac{1}{2}(1+\hat{\rho})}}\right)$	$\Theta\left(\frac{1}{n^{\frac{1}{2}(2+\hat{\rho})}}\right)$	$o\left(\frac{1}{n^{\frac{1}{2}(1+\hat{\rho})}}\right)$
$R^* = I_s(Q)$	$\rightarrow 0$	$\rightarrow 0$	1	$\omega\left(\frac{1}{\sqrt{n}}\right)$	$O\left(\frac{1}{\sqrt{n}}\right)$	$o\left(\frac{1}{\sqrt{n}}\right)$
$R^* > I_s(Q)$	0	$> 0$	1	$\Theta(1)$	$\Theta\left(\frac{1}{\sqrt{n}}\right)$	$o\left(\frac{1}{\sqrt{n}}\right)$

where  $\tilde{P}_s^n(\mathbf{x}|\mathbf{y}) \triangleq \prod_{i=1}^n \tilde{P}_s(x_i|y_i)$ . We write the empirical distribution of  $\mathbf{y}$  as  $\hat{P}_\mathbf{y}$ , and we let  $P_Y$  denote the PMF of  $Y$ .

**Lemma 2.** *Let  $s > 0$  and  $\hat{\rho} \in [0, 1]$  be given. If the pair  $(Q, W)$  is non-singular, then the set*

$$\mathcal{F}_{\hat{\rho},s}^n(\delta) \triangleq \left\{ \mathbf{y} : P_Y(\mathbf{y}) > 0, \max_{\mathbf{y}} |\hat{P}_\mathbf{y}(y) - P_{\hat{\rho},s}^*(y)| \leq \delta \right\} \quad (53)$$

satisfies the following properties:

1) For any  $\mathbf{y} \in \mathcal{F}_{\hat{\rho},s}^n(\delta)$ , we have

$$\text{Var}[i_s^n(\mathbf{X}_s, \mathbf{Y}) | \mathbf{Y} = \mathbf{y}] \geq n(c_3 - r(\delta)), \quad (54)$$

where  $r(\delta) \rightarrow 0$  as  $\delta \rightarrow 0$ .

2) For any  $\delta > 0$ , we have

$$\liminf_{n \rightarrow \infty} -\frac{1}{n} \log \frac{\sum_{\mathbf{x}, \mathbf{y} \notin \mathcal{F}_{\hat{\rho},s}^n(\delta)} Q^n(\mathbf{x}) W^n(\mathbf{y}|\mathbf{x}) e^{-\hat{\rho} i_s^n(\mathbf{x}, \mathbf{y})}}{e^{-nE_0(Q, \hat{\rho}, s)}} > 0. \quad (55)$$

*Proof:* This is a simple refinement of [9, Lemma 3]. ■

Since the two statements of Lemma 2 hold true for any  $\hat{\rho} \in [0, 1]$ , they also hold true when  $\hat{\rho}$  varies within this range, thus allowing us to handle rates which vary with  $n$ .

By upper bounding  $M - 1$  by  $M$  in (1), we obtain

$$\text{rcu}(n, M) \leq S_0(\hat{\rho}, s, \delta) + \sum_{\mathbf{x}, \mathbf{y} \in \mathcal{F}_{\hat{\rho},s}^n(\delta)} Q^n(\mathbf{x}) W^n(\mathbf{y}|\mathbf{x}) \times \min \left\{ 1, M \mathbb{P}[i_s^n(\bar{\mathbf{X}}, \mathbf{y}) \geq i_s^n(\mathbf{x}, \mathbf{y})] \right\}, \quad (56)$$

where  $S_0(\hat{\rho}, s, \delta)$  is a sum of the same form as the second term in (56) with  $\mathbf{y} \notin \mathcal{F}_{\hat{\rho},s}^n(\delta)$ , and we have replaced  $W^n$  by  $i_s^n$  since each is an increasing function of the other. Following [7, Sec. 3.4.5], we have the following when  $\tilde{P}_s^n(\bar{\mathbf{x}}, \mathbf{y}) \neq 0$ :

$$Q^n(\bar{\mathbf{x}}) = Q^n(\bar{\mathbf{x}}) \frac{\tilde{P}_s^n(\bar{\mathbf{x}}|\mathbf{y})}{\tilde{P}_s^n(\bar{\mathbf{x}}|\mathbf{y})} = \tilde{P}_s^n(\bar{\mathbf{x}}|\mathbf{y}) e^{-i_s^n(\bar{\mathbf{x}}, \mathbf{y})}. \quad (57)$$

Summing (57) over all  $\bar{\mathbf{x}}$  such that  $i_s^n(\bar{\mathbf{x}}, \mathbf{y}) \geq t$  yields

$$\mathbb{P}[i_s^n(\bar{\mathbf{X}}, \mathbf{y}) \geq t] = \mathbb{E} \left[ e^{-i_s^n(\mathbf{X}_s, \mathbf{Y})} \mathbb{1} \{ i_s^n(\mathbf{X}_s, \mathbf{Y}) \geq t \} \mid \mathbf{Y} = \mathbf{y} \right] \quad (58)$$

under the joint distribution in (52).

We now observe that (58) is of the same form as the left-hand side of (51). We apply Lemma 1 with  $Q_k$  given by the

PMFs of  $i_s(X_s, y)$  under  $X_s \sim \tilde{P}_s(\cdot|y)$  for the various  $y$  values. We have from (51), (54) and (58) that

$$\mathbb{P}[i_s^n(\bar{\mathbf{X}}, \mathbf{y}) \geq t] \leq \frac{\psi_s}{\sqrt{2\pi n}(c_3 - r(\delta))} e^{-t} (1 + o(1)) \quad (59)$$

for all  $\mathbf{y} \in \mathcal{F}_{\hat{\rho},s}^n(\delta)$  and sufficiently small  $\delta$  (recall that  $c_3 > 0$ ). Here we have used the fact that  $\psi_0$  in (50) coincides with  $\psi_s$  in (22), which follows from (20) and the fact that  $\tilde{P}_s(x|y) > 0$  if and only if  $Q(x)W(y|x) > 0$  (see (17)).

Using the uniformity of the  $o(1)$  term in  $t$  in (59) (see Lemma 1), taking  $\delta \rightarrow 0$  (and hence  $r(\delta) \rightarrow 0$ ), and writing

$$\min\{1, f_n(1 + \zeta_n)\} \leq (1 + |\zeta_n|) \min\{1, f_n\}, \quad (60)$$

we see that the second term in (56) is upper bounded by  $\text{rcu}_s^*(n, M)(1 + o(1))$ . Finally, using (55) (along with (23) and (33)), it is easily shown that  $S_0(\hat{\rho}, s, \delta)$  can be factored into the  $1 + o(1)$  term, thus completing the proof of (31).

## REFERENCES

- [1] R. Gallager, *Information Theory and Reliable Communication*. John Wiley & Sons, 1968.
- [2] V. Strassen, "Asymptotische Abschätzungen in Shannon's Informations-theorie," in *Trans. 3rd Prague Conf. on Inf. Theory*, 1962, pp. 689–723, [English Translation: <http://www.math.wustl.edu/~luthy/strassen.pdf>].
- [3] Y. Polyanskiy, V. Poor, and S. Verdú, "Channel coding rate in the finite blocklength regime," *IEEE Trans. Inf. Theory*, vol. 56, no. 5, pp. 2307–2359, May 2010.
- [4] Y. Altuğ and A. B. Wagner, "Moderate deviations in channel coding," 2012, <http://arxiv.org/abs/1208.1924>.
- [5] A. Martinez and A. Guillén i Fàbregas, "Saddlepoint approximation of random-coding bounds," in *Inf. Theory App. Workshop*, La Jolla, CA, 2011.
- [6] J. L. Jensen, *Saddlepoint Approximations*. Oxford University Press, 1995.
- [7] Y. Polyanskiy, "Channel coding: Non-asymptotic fundamental limits," Ph.D. dissertation, Princeton University, 2010.
- [8] Y. Altuğ and A. B. Wagner, "Refinement of the random coding bound," 2014, <http://arxiv.org/abs/1312.6875>.
- [9] J. Scarlett, A. Martinez, and A. Guillén i Fàbregas, "A derivation of the asymptotic random-coding prefactor," in *Allerton Conf. on Comm., Control and Comp.*, Monticello, IL, 2013.
- [10] —, "Mismatched decoding: Error exponents, second-order rates and saddlepoint approximations," 2014, *IEEE Trans. Inf. Theory*, to appear [Online: <http://arxiv.org/abs/1303.6166>].
- [11] W. Feller, *An introduction to probability theory and its applications*, 2nd ed. John Wiley & Sons, 1971, vol. 2.
- [12] J. Mineka and S. Silverman, "A local limit theorem and recurrence conditions for sums of independent non-lattice random variables," *Annals Math. Stats.*, vol. 41, no. 2, pp. 592–600, April 1970.
- [13] V. V. Petrov, *Sums of Independent Random Variables*. Springer-Verlag, 1975.