A Derivation of the Cost-Constrained Sphere-Packing Exponent

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***Abstract*—We derive the channel-coding sphere-packing exponent under a per-codeword cost constraint. The proof is based on hypothesis testing and holds for continuous memoryless channels.**

***Index Terms*—Channel coding, reliability function, sphere-packing exponent, cost constraint, continuous channel.**

I. Introduction

The behavior of the channel-coding error probability may be quantified in terms of error exponents, defined as the rate of the error probability's exponential decay in the block length [1], [2]. Lower bounds on the exponent for discrete memoryless channels (DMC) are easily obtained by random-coding techniques. In contrast, the computation of upper bounds, satisfied by every code, is more challenging since code-specific bounds need to be optimized over each possible codebook. Nevertheless, certain bounds avoid this optimization, e. g. the sphere-packing bound [3], which is exponentially tight for rates above the critical rate of the channel [1], [3].

The sphere-packing exponent has been derived using different techniques. By building on an instance of binary hypothesis testing, Shannon, Gallager and Berlekamp [3] derived an error bound with the sphere-packing exponent (SP67); also based on hypothesis testing, Blahut proposed an alternative derivation of this bound in [4]; the sphere-packing exponent was also obtained by using combinatorial methods in [5]; and based on the method of types in [2]. The works [6], [7] addressed the tightness of the SP67 bound for short to moderate block lengths by improving the pre-exponential and rate penalty terms. Recently, the metaconverse bound [8] has been shown to have the exponential decay of the sphere-packing bound [9].

Cost constraints were first included in the derivation of the sphere-packing bound in [10], by using a geometric approach for the specific case of the Gaussian channel. The SP67 [3] can also be extended to introduce cost constraints in general memoryless semicontinuous channels [1, p. 329, footnote]. In this work, we generalize the derivation of the sphere-packing exponent in [4] to consider per-codeword cost constraints. In contrast to the derivation in [3], no assumption of constant-composition codewords is needed. This allows to extend the cost-constrained sphere-packing exponent to arbitrary continuous memoryless channels. Building on this result, we establish a connection between the cost-constrained sphere-packing and Csiszár sphere-packing exponent for constant composition codes [2, Ch. 2, Th. 5.3].

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II. Preliminaries

We study the problem of transmitting M equiprobable messages over a DMC using length-n block codes. The channel law is given by $W^n(\boldsymbol{y}|\boldsymbol{x}) = \prod_{i=1}^n W(y_i|x_i)$, $\boldsymbol{x} = (x_1, \ldots, x_n)$, $x \in \mathcal{X}$, $\boldsymbol{y} = (y_1, \ldots, y_n)$, $y \in \mathcal{Y}$. We define a separable cost function $f_n(\boldsymbol{x}) \triangleq \sum_{i=1}^n f(x_i)$ with $f(x)$ denoting a real-valued scalar cost. A cost-constrained codebook $\mathcal{C}$ is defined as a set of codewords $\{\boldsymbol{x}_m\}_{m=1}^M$ such that $f_n(\boldsymbol{x}_m) \leq n\xi$, $m = 1, \ldots, M$, where ξ is the per-symbol cost cap. The coding rate is $R \triangleq \frac{1}{n} \log M$.

An encoder maps the source message $m \in \{1, \ldots, M\}$ to a length-n codeword $\boldsymbol{x}_m$, which is then transmitted over the channel. The channel output $\boldsymbol{y}$ is decoded at the receiver following a maximum likelihood (ML) criterium. Let us denote the output of the decoder as $\hat{m}(\boldsymbol{y})$. Then, the error probability incurred when a message m was transmitted is

$$\epsilon_m(\mathcal{C}) \triangleq \Pr\{\hat{m}(\boldsymbol{Y}) \neq m\}, \tag{1}$$

and the error probability averaged over all codewords is thus

$$\epsilon(\mathcal{C}) \triangleq \frac{1}{M} \sum_{m=1}^{M} \epsilon_m(\mathcal{C}). \tag{2}$$

Similarly, we define the maximal error probability as $\epsilon_{\max}(\mathcal{C}) \triangleq \max_m \epsilon_m(\mathcal{C})$.

We say that an error exponent $E > 0$ is *achievable* if there exists a sequence of codes $\mathcal{C}_n$, $n = 1, 2, \ldots$, such that the average error probability $\epsilon(\mathcal{C}_n)$ is upper-bounded as

$$\epsilon(\mathcal{C}_n) \leq e^{-nE + o(n)}, \tag{3}$$

where $o(n)$ satisfies $\lim_{n\to\infty} o(n)/n = 0$.

Any achievable error exponent for a DMC is upper bounded [2]–[5] as $E \leq E_{\text{sp}}(R - \delta)$, for any $\delta > 0$, where the sphere-packing exponent $E_{\text{sp}}(R)$ is given by

$$E_{\text{sp}}(R) \triangleq \sup_{\rho \geq 0} \big\{ E_0(\rho) - \rho R \big\}. \tag{4}$$

Gallager's E_0 function is $E_0(\rho) \triangleq \max_P E_0(\rho, P)$, with

$$E_0(\rho, P) \triangleq -\log \sum_y \left(\sum_x P(x) W(y|x)^{\frac{1}{1+\rho}} \right)^{1+\rho}. \quad (5)$$

The error exponent achievable by a sequence of cost constrained codebooks is upper bounded as $E \leq E_{\text{sp}}^{\text{cost}}(R-\delta)$, for arbitrary $\delta > 0$, where

$$E_{\text{sp}}^{\text{cost}}(R) \triangleq \sup_{\rho \geq 0} \big\{ E_0^{\text{cost}}(\rho) - \rho R \big\}, \quad (6)$$

with $E_0^{\text{cost}}(\rho) \triangleq \max_{P \in \mathcal{P}_\xi, s \geq 0} E_0^{\text{cost}}(\rho, P, s)$, $\mathcal{P}_\xi$ being the set of input distributions satisfying the cost constraint, and

$$E_0^{\text{cost}}(\rho, P, s) \triangleq -\log \sum_y \left(\sum_x P(x) e^{s(f(x)-\xi)} W(y|x)^{\frac{1}{1+\rho}} \right)^{1+\rho}. \quad (7)$$

A. Hypothesis Testing

Based on an observation $v \in \mathcal{V}$ in some alphabet $\mathcal{V}$, consider a binary hypothesis test between the hypotheses

$$\mathcal{H}_0: \quad V \sim P_0, \quad (8)$$
$$\mathcal{H}_1: \quad V \sim P_1, \quad (9)$$

where P_0 and P_1 are distributions over $\mathcal{V}$. For a binary hypothesis test $T: \mathcal{V} \to \{\mathcal{H}_0, \mathcal{H}_1\}$ we define the type-I error as deciding $\mathcal{H}_1$ when the true hypothesis is $\mathcal{H}_0$; and the type-II error as deciding $\mathcal{H}_0$ when the true hypothesis is $\mathcal{H}_1$. These error probabilities are respectively given by

$$\epsilon_{\text{I}}(T) = \mathbb{P}_0\big\{T(V) = \mathcal{H}_1\big\}, \quad (10)$$
$$\epsilon_{\text{II}}(T) = \mathbb{P}_1\big\{T(V) = \mathcal{H}_0\big\}, \quad (11)$$

where $\mathbb{P}_0\{\mathcal{E}\}$ and $\mathbb{P}_1\{\mathcal{E}\}$ denote the probability of the event $\mathcal{E}$ computed with respect to P_0 and P_1, respectively. We define the smallest type-I error among all (possibly randomized) tests T with a type-II error at most β as

$$\alpha_\beta(P_0, P_1) \triangleq \min_{T: \epsilon_{\text{II}}(T) \leq \beta} \epsilon_{\text{I}}(T). \quad (12)$$

A bound on the exponential behavior of lowest type-I and type-II errors was found by Blahut in [4, Th. 10]. Let us define

$$e(r) \triangleq \sup_{\rho \geq 0} \left\{ -\rho r - \log \left(\sum_v P_0(v)^{\frac{1}{1+\rho}} P_1(v)^{\frac{\rho}{1+\rho}} \right)^{1+\rho} \right\}, \quad (13)$$

and, for $\hat{\rho}$ maximizing (13) and for every v, let us define

$$\hat{p}(v) \triangleq \frac{P_0(v)^{\frac{1}{1+\hat{\rho}}} P_1(v)^{\frac{\hat{\rho}}{1+\hat{\rho}}}}{\sum_v P_0(v)^{\frac{1}{1+\hat{\rho}}} P_1(v)^{\frac{\hat{\rho}}{1+\hat{\rho}}}}, \quad (14)$$

Theorem 1 ([4, Th. 10]): Let $\nu > 0$ be given, and let $\zeta \in (0,1)$ be arbitrary. For any $\beta \leq \zeta e^{-(r+\nu)}$ we have that

$$\alpha_\beta(P_0, P_1) \geq \left(1 - \frac{\sigma_0^2 + \sigma_1^2}{\nu^2} - \zeta\right) e^{-(e(r)+\nu)}, \quad (15)$$

where σ_i^2 denotes the variance of the random variable $\log \frac{\hat{p}(V)}{P_i(V)}$ with respect to the distribution $\hat{p}$, $i = 0, 1$.

III. Cost-Constrained Sphere-Packing

For each message $m = 1, \ldots, M$, and based on the channel output $\boldsymbol{y}$, we define a binary hypothesis test between $P_0 = W^n(\cdot|\boldsymbol{x}_m)$ and $P_1 = Q^n$, where Q^n is a distribution over $\mathcal{Y}^n$ independent of m. Consider a (possibly suboptimal) bank of tests $\{T_m\}$ defined as follows. Based on the channel decoder, the test T_m decides $\mathcal{H}_0$ if $\hat{m}(\boldsymbol{y}) = m$, and $\mathcal{H}_1$ otherwise. For any partition on the output space induced by $\hat{m}(\boldsymbol{y})$, it holds

$$\sum_{m=1}^{M} \epsilon_{\text{II}}(T_m) = \sum_{\boldsymbol{y}} Q^n(\boldsymbol{y}) = 1. \quad (16)$$

Then, since $\epsilon_{\text{II}}(T_m) \geq 0$, $m = 1 \ldots, M$, there must exist at least one message m such that $\epsilon_{\text{II}}(T_m) \leq \frac{1}{M}$. In the remainder of this paper, we fix m such that $\epsilon_{\text{II}}(T_m) \leq \frac{1}{M}$.

The error probability of this message m is

$$\epsilon_m(\mathcal{C}) = \Pr\{\hat{m}(\boldsymbol{Y}) \neq m\} = \epsilon_{\text{I}}(T_m). \quad (17)$$

As (12) is a lower bound for any test, and using that $\epsilon_{\text{II}}(T_m) \leq \frac{1}{M}$, the maximal error probability can be lower bounded as

$$\epsilon_{\max}(\mathcal{C}) \;\geq\; \epsilon_m(\mathcal{C}) \;\geq\; \alpha_{\frac{1}{M}}\big(W^n(\cdot|\boldsymbol{x}_m), Q^n\big). \quad (18)$$

We now bound the exponential behavior of (18). We define

$$\Lambda_n(\rho, Q^n, \boldsymbol{x}) \triangleq -\frac{1}{n} \log \left(\sum_{\boldsymbol{y}} W^n(\boldsymbol{y}|\boldsymbol{x})^{\frac{1}{1+\rho}} Q^n(\boldsymbol{y})^{\frac{\rho}{1+\rho}} \right)^{1+\rho}, \quad (19)$$

and the sequences

$$\nu_n' \triangleq \nu_n'' + n^{-1} \log \zeta, \quad (20)$$
$$\nu_n'' \triangleq n^{\alpha-1}, \quad 1/2 < \alpha < 1. \quad (21)$$

For sufficiently large n, $\nu_n' > 0$. Then, we apply Theorem 1 with $\nu = n\nu_n'$, $P_0 = W^n(\cdot|\boldsymbol{x}_m)$, $P_1 = Q^n$, and $r = nR - \nu + \log \zeta$. Since $\beta = \zeta e^{-(r+\nu)} = \frac{1}{M}$, Theorem 1 yields

$$E_{\max} \triangleq \lim_{n\to\infty} -\frac{1}{n} \log \epsilon_{\max}(\mathcal{C}_n) \quad (22)$$
$$\leq \lim_{n\to\infty} \sup_{\rho \geq 0} \big\{ \Lambda_n(\rho, Q^n, \boldsymbol{x}_m) - \rho\,(R - \nu_n'') \big\}$$
$$+ \lim_{n\to\infty} \left(\nu_n' - \frac{1}{n} \log \left(1 - \frac{\sigma_0^2 + \sigma_1^2}{(n\nu_n')^2} - \zeta \right) \right). \quad (23)$$

The second limit in (23) vanishes since, for ν_n' in (20), $\lim_{n\to\infty} \nu_n' = 0$, and $\lim_{n\to\infty} \frac{\sigma_i^2}{(n\nu_n')^2} = \lim_{n\to\infty} \frac{\sigma_i^2}{n^{2\alpha}} = 0$, since the variances σ_i^2, $i = 0, 1$, are proportional to n, and $2\alpha > 1$. Then,

$$E_{\max} \leq \lim_{n\to\infty} \sup_{\rho \geq 0} \big\{ \Lambda_n(\rho, Q^n, \boldsymbol{x}_m) - \rho\,(R - \nu_n'') \big\}. \quad (24)$$

For each n, we choose Q^n such that we obtain the lowest upper bound in (24). We next show that, for an arbitrary $\delta > 0$,

$$E_{\max} \leq \lim_{n\to\infty} \inf_{Q^n} \sup_{\rho \geq 0} \big\{ \Lambda_n(\rho, Q^n, \boldsymbol{x}_m) - \rho\,(R - \nu_n'') \big\} \quad (25)$$
$$\leq \lim_{n\to\infty} \sup_{\rho \geq 0} \inf_{Q^n} \big\{ \Lambda_n(\rho, Q^n, \boldsymbol{x}_m) - \rho(R - \delta) \big\}, \quad (26)$$

where Q^n is not allowed to depend on $\boldsymbol{x}_m$.

In order to show (26), assume first that the value of ρ achieving the saddlepoint in (25) is finite. Then, there exists $\bar{\rho} < \infty$ such that

$$E_{\max} \leq \lim_{n\to\infty} \inf_{Q^n} \sup_{\rho\geq 0} \left\{\Lambda_n(\rho, Q^n, \boldsymbol{x}_m) - \rho\left(R - \nu_n''\right)\right\} \quad (27)$$

$$= \lim_{n\to\infty} \inf_{Q^n} \max_{0\leq\rho\leq\bar{\rho}} \left\{\Lambda_n(\rho, Q^n, \boldsymbol{x}_m) - \rho\left(R - \nu_n''\right)\right\} \quad (28)$$

$$= \lim_{n\to\infty} \sup_{0\leq\rho\leq\bar{\rho}} \inf_{Q^n} \left\{\Lambda_n(\rho, Q^n, \boldsymbol{x}_m) - \rho\left(R - \nu_n''\right)\right\}, \quad (29)$$

where in (28) we used that the saddle point is achieved at $\rho < \bar{\rho}$; and (29) follows from the Kneser-Fan minimax theorem [11, Th. 4.2], since, for fixed Q^n, the bracketed term in (28) is concave in ρ, and, for fixed ρ it is convex in Q^n. Then, (26) follows from (29) by increasing the range over which the maximization over ρ is performed, and by using that, for arbitrary $\delta > 0$ and sufficiently large n, $\nu_n'' \leq \delta$.

When the saddle point in (25) is attained at $\rho \to \infty$, we cannot apply Kneser-Fan minimax theorem. Let $\Lambda_n'(\cdot)$ denote the derivative of $\Lambda_n(\cdot)$ with respect to ρ. Since the optimizer in (25) is $\rho \to \infty$, it follows that $\Lambda_n'(\rho, Q^n, \boldsymbol{x}_m) \geq R - \nu_n''$ for all Q^n, $\rho \geq 0$. Using that, for sufficiently large n, $\Lambda_n'(\rho, Q^n, \boldsymbol{x}_m) \geq R - \nu_n'' \geq R - \delta$, the bound (26) becomes $E_{\max} \leq \infty$, which is trivially true. Then, (26) holds regardless the value of the optimizing ρ.

The dependence on the sequence of codebooks is present in (26) through $\Lambda_n(\rho, Q^n, \boldsymbol{x}_m)$. This dependence is circumvented by making use of the following property [1, Thm. 5.6.5].

Theorem 2: For $\rho \geq 0$, let

$$\mu_0(\boldsymbol{y}, \rho) \triangleq \sum_{\boldsymbol{x}} \widehat{P}^n(\boldsymbol{x}) W^n(\boldsymbol{y}|\boldsymbol{x})^{\frac{1}{1+\rho}}, \quad (30)$$

where $\widehat{P}^n$ is an exponent-achieving distribution, i. e., $\widehat{P}^n(\boldsymbol{x}) = \prod_{i=1}^n \widehat{P}(x_i)$, $\widehat{P} = \arg\max_P \left\{E_0\left(\rho, P\right)\right\}$.

Then, for any $\rho \geq 0$ and any $\boldsymbol{x}$, it holds that

$$\sum_{\boldsymbol{y}} W^n(\boldsymbol{y}|\boldsymbol{x})^{\frac{1}{1+\rho}} \mu_0(\boldsymbol{y}, \rho)^{\rho} \;\geq\; \sum_{\boldsymbol{y}} \mu_0(\boldsymbol{y}, \rho)^{1+\rho}. \quad (31)$$

In (31), only the left-hand side depends on $\boldsymbol{x}$. We define

$$Q_{0,\rho}^n(\boldsymbol{y}) \triangleq \frac{\mu_0(\boldsymbol{y}, \rho)^{1+\rho}}{\sum_{\boldsymbol{y}} \mu_0(\boldsymbol{y}, \rho)^{1+\rho}}. \quad (32)$$

Eq. (32) corresponds to that in [4, Cor. 4] (see also [2, p. 193, Prob. 23], [3, Eq. (4.20)]). Using Theorem 2 it can be verified that $\Lambda_n\big(\rho, Q_{0,\rho}^n, \boldsymbol{x}_m\big) \leq E_0(\rho)$ for every code in the sequence. Hence, by letting $Q^n = Q_{0,\rho}^n$ in (26) we obtain

$$E_{\max} \leq \sup_{\rho\geq 0} \big\{E_0(\rho) - \rho(R-\delta)\big\} = E_{\text{sp}}(R-\delta). \quad (33)$$

In order to introduce a cost constraint into this formulation we make use of the following extension of Theorem 2.

Theorem 3: For $\rho \geq 0$, let

$$\mu_1(\boldsymbol{y}, \rho) \triangleq \sum_{\boldsymbol{x}} \widehat{P}^n(\boldsymbol{x}) e^{\hat{s}(f_n(\boldsymbol{x}) - n\xi)} W^n(\boldsymbol{y}|\boldsymbol{x})^{\frac{1}{1+\rho}}, \quad (34)$$

where $\widehat{P}^n(\boldsymbol{x}) = \prod_{i=1}^n \widehat{P}(x_i)$, and

$$\{\widehat{P}, \hat{s}\} = \arg \max_{P\in\mathcal{P}_\xi, s\geq 0} E_1(\rho, P, s). \quad (35)$$

For any $\boldsymbol{x}$ such that $f_n(\boldsymbol{x}) \leq n\xi$, it holds that

$$\sum_{\boldsymbol{y}} W^n(\boldsymbol{y}|\boldsymbol{x})^{\frac{1}{1+\rho}} \mu_1(\boldsymbol{y}, \rho)^{\rho} \;\geq\; \sum_{\boldsymbol{y}} \mu_1(\boldsymbol{y}, \rho)^{1+\rho}. \quad (36)$$

Proof: See Appendix A. ∎

We define

$$Q_{1,\rho}^n(\boldsymbol{y}) = \frac{\mu_1(\boldsymbol{y}, \rho)^{1+\rho}}{\sum_{\boldsymbol{y}'} \mu_1(\boldsymbol{y}', \rho)^{1+\rho}}. \quad (37)$$

Substituting $Q^n = Q_{1,\rho}^n$ in (19) yields

$$\Lambda_n\big(\rho, Q_{1,\rho}^n, \boldsymbol{x}_m\big) \leq -\frac{1}{n} \log\left(\left(\sum_{\boldsymbol{y}} \mu_1(\boldsymbol{y}, \rho)^{1+\rho}\right)^{\frac{1}{1+\rho}}\right)^{1+\rho} \quad (38)$$

$$= E_0^{\text{cost}}(\rho). \quad (39)$$

where in (38) we applied Theorem 3 for $\boldsymbol{x}_m$ satisfying the cost constraint; and (39) follows from the definition of $\mu_1(\boldsymbol{y}, \rho)$ and the E_0^{cost} function.

Hence, from (26) and (38)-(39), we obtain

$$E_{\max} \leq \sup_{\rho\geq 0} \big\{E_0^{\text{cost}}(\rho) - \rho(R-\delta)\big\} = E_{\text{sp}}^{\text{cost}}(R-\delta). \quad (40)$$

For any code, $\epsilon(\mathcal{C}_n) \geq \frac{1}{2}\epsilon_{\max}(\mathcal{C}_n')$ where $\mathcal{C}_n'$ is an expurgated code obtained by removing from $\mathcal{C}_n$ the $M/2$ codewords with highest error probability [4, Th. 20]. As the rate R is unaffected by the expurgation, combining (33) and (40) yields the following result.

Theorem 4: For a memoryless channel W^n, let E denote the error exponent achievable by a sequence of codebooks $\mathcal{C}_n$, $n = 1, 2, \ldots$, such that, for each value of n, the codewords satisfy an individual (separable) cost constraint $f_n(\boldsymbol{x}_m) \leq n\xi$, $m = 1, \ldots, M$. It follows that, for any $\delta > 0$,

$$E \leq \sup_{\rho\geq 0} \big\{\min\big(E_0(\rho), E_0^{\text{cost}}(\rho)\big) - \rho(R-\delta)\big\}. \quad (41)$$

If the cost constraint is active for any P achieving $E_1(\rho)$, then $E_1(\rho) \leq E_0(\rho)$. However, if the cost constraint is non-active, $E_0(\rho) \leq E_1(\rho)$. Therefore, neither $E_{\text{sp}}(R)$ nor $E_{\text{sp}}^{\text{cost}}(R)$ dominates in general.

This theorem applies to codebooks satisfying a per-codeword cost constraint. Extending Theorem 4 to codebooks satisfying an average cost constraint is still an open problem.

IV. Connection with Constant Composition Codes

For a given n, consider a constant composition code with empirical distribution P_n, i. e., every codeword $\boldsymbol{x}$ belonging to $\mathcal{C}_n$ has a composition equal to P_n. We fix Q^n to be an arbitrary product distribution, $Q^n(\boldsymbol{y}) = \prod_{i=1}^n Q(y_i)$. In this case $\Lambda_n\big(\rho, Q^n, \boldsymbol{x}\big)$ is independent of the specific code,

$$\Lambda_n\big(\rho, Q^n, \boldsymbol{x}\big) = -\sum_x \mathsf{P}_n(x) \log\left(\sum_y W(y|x)^{\frac{1}{1+\rho}} Q(y)^{\frac{\rho}{1+\rho}}\right)^{1+\rho}. \quad (42)$$

For any sequence of constant composition codes such that $\mathsf{P}_n \to \mathsf{P}$ as $n \to \infty$, from (26) and (42) it follows that

$$E_{\max} \leq \sup_{\rho \geq 0} \big\{ E_1(\rho, \mathsf{P}) - \rho(R-\delta) \big\}, \tag{43}$$

where

$$E_1(\rho, P) \triangleq \min_{Q} \left\{ -\sum_x P(x) \log \left(\sum_y W(y|x)^{\frac{1}{1+\rho}} Q(y)^{\frac{\rho}{1+\rho}} \right)^{1+\rho} \right\}. \tag{44}$$

Eq. (43) corresponds to Csiszár sphere-packing bound for constant composition codes [2, Ch. 2, Th. 5.3]. Optimizing (43) over compositions that satisfy the cost constraint and applying the expurgation argument, we obtain

$$E \leq \sup_{\rho \geq 0} \big\{ E_1(\rho) - \rho(R-\delta) \big\}, \tag{45}$$

where $E_1(\rho) \triangleq \max_{P \in \mathcal{P}_\xi} E_1(\rho, P)$.

We next show that (45) coincides with (41) in Theorem 4. To this end, let us consider the dual formulation of $E_1(\rho)$, which is given by a double optimization over distributions P satisfying the cost constraint, and over functions $a : \mathcal{X} \to \mathbb{R}$ with finite average $\bar{a} \triangleq \sum_x P(x)a(x)$ [12, Th. 3.4]:

$$E_1(\rho) = \max_{P \in \mathcal{P}_\xi, a} E_1(\rho, P, a), \tag{46}$$

$$E_1(\rho, P, a) \triangleq -\log \sum_y \left(\sum_x P(x) e^{a(x)-\bar{a}} W(y|x)^{\frac{1}{1+\rho}} \right)^{1+\rho}. \tag{47}$$

Note the similarity between $E_0^{\text{cost}}(\rho, P, s)$ in (7) and (47). While $a(x)$ in the definition of $E_1(\rho, P, a)$ in (47) is an arbitrary function to be optimized, $f(x)$ in (7) denotes the cost function, which is given.

Appendix B derives the optimality conditions for the optimization problem in (46). Let us define $P_0 \triangleq \arg\max_P E_0(\rho, P)$. When $P_0 \in \mathcal{P}_\xi$, the cost constraint is not active in (46), and the maximum is attained for $a(x) = \bar{a}$, $\forall x$. Hence, in this case $E_1(\rho, P, a)$ becomes $E_0(\rho, P)$, and $E_1(\rho) = E_0(\rho)$. In contrast, for $P_0 \notin \mathcal{P}_\xi$, the optimizing $a(\cdot)$ is $a(x) = sf(x)$, $\forall x$, for some $s \geq 0$. Using that the cost constraint holds with equality in this case, we obtain $E_1(\rho) = E_0^{\text{cost}}(\rho)$. By combining both possibilities, (46) yields $E_1(\rho) = \min\big\{E_0(\rho), E_0^{\text{cost}}(\rho)\big\}$ and (45) coincides with (41).

Appendix A
Proof of Theorem 3

Let $\rho \geq 0$ be fixed. Let us define

$$\Phi_y(P, s) \triangleq \sum_x P(x) e^{s(f(x)-\xi)} W(y|x)^{\frac{1}{1+\rho}}. \tag{48}$$

$$\Psi_y(P, s) \triangleq \sum_x P(x)(f(x)-\xi) e^{s(f(x)-\xi)} W(y|x)^{\frac{1}{1+\rho}}. \tag{49}$$

We study the optimality conditions of the following optimization problem, which is equivalent to $\max_{P \in \mathcal{P}_\xi, s \geq 0} E_1(\rho, P, s)$,

$$\begin{aligned} \min_{P,s} \quad & \sum_y \Phi_y(P, s)^{1+\rho}, \\ \text{subject to} \quad & s \geq 0,\ P(x) \geq 0, \\ & \sum_x P(x) = 1, \\ & \sum_x P(x) f(x) \leq \xi. \end{aligned} \tag{50}$$

The Lagrangian of the optimization problem in (50) is

$$\begin{aligned} \mathcal{L}(P, s) = & \sum_y \Phi_y(P, s)^{1+\rho} - \sigma s - \sum_x \eta_x P(x) \\ & - \lambda \Big(\sum_x P(x) - 1 \Big) - \gamma \sum_x P(x) \big(\xi - f(x) \big), \end{aligned} \tag{51}$$

where $\sigma \geq 0$, $\eta_{\boldsymbol{x}} \geq 0$, $\lambda \in \mathbb{R}$ and $\gamma \geq 0$ are the Lagrange multipliers associated to the respective constraints in (50).

Let us denote by $\widehat{P}$, $\hat{s}$ the values of P, s optimizing (50). Similarly, let us define $\widehat{\Phi}_y \triangleq \Phi_y\big(\widehat{P}, \hat{s}\big)$, $\widehat{\Psi}_y \triangleq \Psi_y\big(\widehat{P}, \hat{s}\big)$. By taking the derivative of (51) with respect to $P(x)$ and equating it to zero we obtain the following optimality condition

$$\begin{aligned} (1+\rho) \sum_y & e^{\hat{s}(f(x)-\xi)} W(y|x)^{\frac{1}{1+\rho}} \big(\widehat{\Phi}_y \big)^{\rho} \\ & = \eta_x + \lambda + \gamma \big(\xi - f(x) \big). \end{aligned} \tag{52}$$

Likewise, by taking the derivative of (51) with respect to s and equating it to zero yields the condition

$$(1+\rho) \sum_y \widehat{\Psi}_y \big(\widehat{\Phi}_y \big)^{\rho} = \sigma. \tag{53}$$

Multiplying both sides of (52) by $\widehat{P}(x)$, summing over x, and using that due to complementary slackness [13, Sec. 5.5.2], $\eta_x \widehat{P}(x) = 0$, $\gamma \sum_x \widehat{P}(x) \big(\xi - f(x) \big) = 0$, we obtain

$$\lambda = (1+\rho) \sum_y \big(\widehat{\Phi}_y \big)^{1+\rho}. \tag{54}$$

Multiplying (52) by $\widehat{P}(x)(f(x)-\xi)$, summing over x, yields

$$\begin{aligned} (1+\rho) \sum_y \widehat{\Psi}_y \big(\widehat{\Phi}_y \big)^{\rho} = & \sum_x \eta_x \widehat{P}(x) \big(f(x) - \xi \big) \\ & + \lambda \sum_x \widehat{P}(x) \big(f(x) - \xi \big) - \gamma \sum_x \widehat{P}(x) \big(f(x) - \xi \big)^2. \end{aligned} \tag{55}$$

Substituting (53) in (55), using that $\eta_x \widehat{P}(x) = 0$, we obtain

$$\sigma = \lambda \sum_x \widehat{P}(x) \big(f(x) - \xi \big) - \gamma \sum_x \widehat{P}(x) \big(f(x) - \xi \big)^2. \tag{56}$$

If the cost constraint is non-active, i. e., $\sum_x \widehat{P}(x) f(x) < \xi$, the corresponding Lagrange multiplier is $\gamma = 0$ due to complementary slackness. If the cost constraint is active, $\sum_x \widehat{P}(x) f(x) = \xi$, and from (56) we obtain

$$\sigma = -\gamma \sum_x \widehat{P}(x) \big(f(x) - \xi \big)^2. \tag{57}$$

Since $\sigma \geq 0$, $\gamma \geq 0$, from (57) we conclude that $\sigma = \gamma = 0$, so in either case $\gamma = 0$. Substituting (54) into (52), yields

$$\sum_y e^{\hat{s}(f(x)-\xi)} W(y|x)^{\frac{1}{1+\rho}} \big(\widehat{\Phi}_y\big)^{\rho} = \sum_y \big(\widehat{\Phi}_y\big)^{1+\rho} + \frac{\eta_x}{1+\rho}, \tag{58}$$

and since $\eta_x \geq 0$,

$$\sum_y e^{\hat{s}(f(x)-\xi)} W(y|x)^{\frac{1}{1+\rho}} \big(\widehat{\Phi}_y\big)^{\rho} \geq \sum_y (\widehat{\Phi}_y)^{1+\rho}. \tag{59}$$

Finally, we use the definition of $\mu_1(\boldsymbol{y}, \rho)$ in (34) to write

$$\begin{aligned} &\sum_{\boldsymbol{y}} W^n(\boldsymbol{y}|\boldsymbol{x})^{\frac{1}{1+\rho}} \mu_1(\boldsymbol{y}, \rho)^{\rho} \\ &\qquad \geq \sum_{\boldsymbol{y}} e^{\hat{s}(f_n(\boldsymbol{x})-n\xi)} W^n(\boldsymbol{y}|\boldsymbol{x})^{\frac{1}{1+\rho}} \mu_1(\boldsymbol{y}, \rho)^{\rho} \qquad (60) \\ &\qquad \geq \sum_{\boldsymbol{y}} \mu_1(\boldsymbol{y}, \rho)^{1+\rho}, \qquad (61) \end{aligned}$$

where in (60) we used that, by assumption, $f_n(\boldsymbol{x}) \leq n\xi$; and (61) follows from factorizing (60) and applying (59) to each of the factors. The step (60) holds with equality for $f_n(\boldsymbol{x}) = n\xi$, and the step (61) is tight as long as $\widehat{P}^n(\boldsymbol{x}) > 0$.

Appendix B
Optimality conditions for $E_1(\rho)$ in (46)

The Lagrangian of the optimization problem (46) is

$$\begin{aligned} \mathcal{L}(P, a) &= (1+\rho) \sum_x P(x) a(x) \\ &\quad - \log \sum_y \bigg(\sum_x P(x) e^{a(x)} W(y|x)^{\frac{1}{1+\rho}} \bigg)^{1+\rho} \\ &\quad - \lambda \bigg(\sum_x P(x) - 1 \bigg) - \gamma \sum_x P(x) \big(f(x) - \xi \big), \end{aligned} \tag{62}$$

where $\lambda \in \mathbb{R}$ and $\gamma \geq 0$ are the Lagrange multipliers associated to the constraints $\sum_x P(x) = 1$ and $\sum_x P(x) f(x) \leq \xi$, respectively.

Let $\widehat{P}$, $\hat{a}(\cdot)$, denote the values of P, $a(\cdot)$ optimizing (46). Setting the derivative of $\mathcal{L}(P, a)$ with respect to $a(x)$ to zero, we obtain the following optimality condition,

$$\frac{\sum_y e^{\hat{a}(x)} W(y|x)^{\frac{1}{1+\rho}} \Big(\sum_{x'} \widehat{P}(x') e^{\hat{a}(x')} W(y|x')^{\frac{1}{1+\rho}} \Big)^{\rho}}{\sum_y \Big(\sum_{x''} \widehat{P}(x'') e^{\hat{a}(x'')} W(y|x'')^{\frac{1}{1+\rho}} \Big)^{1+\rho}} = 1, \tag{63}$$

for $x \in \mathcal{X}$. Equating to zero the derivative of $\mathcal{L}(P, a)$ with respect to $P(x)$, and using (63) it follows that for $\widehat{P}$, $\hat{a}(\cdot)$,

$$(1+\rho)\big(\hat{a}(x) - 1\big) - \lambda - \gamma\big(f(x) - \xi\big) = 0. \tag{64}$$

Due to complementary slackness, $\gamma \sum_x \widehat{P}(x)\big(f(x) - \xi\big) = 0$. Then, multiplying (64) by $\widehat{P}(x)$, summing over $x \in \mathcal{X}$, yields

$$\lambda = (1+\rho) \sum_x \widehat{P}(x)\big(\hat{a}(x) - 1\big). \tag{65}$$

Substituting (65) in (64), we obtain that, for $x \in \mathcal{X}$,

$$\hat{a}(x) - \sum_{\bar{x}} \widehat{P}(\bar{x}) \hat{a}(\bar{x}) = \frac{\gamma}{1+\rho}\big(f(x) - \xi\big). \tag{66}$$

When the cost constraint is inactive its associated Lagrange multiplier is $\gamma = 0$. Hence, from (66) we obtain that $\hat{a}(x) = \sum_{\bar{x}} \widehat{P}(\bar{x}) \hat{a}(\bar{x})$, $x \in \mathcal{X}$, is a constant. On the other hand, when the cost constraint is active, $\sum_x \widehat{P}(x) f(x) = \xi$ and $\gamma \geq 0$. Then, (66) yields $\hat{a}(x) = s f(x)$ with $s = \frac{\gamma}{1+\rho} \geq 0$.

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